Abstract

Students in a high school calculus class attempt to design the ideal tissue-paper hot-air balloon. This outline provides information on project design, construction of balloons, mathematical analysis of surface area and volume of each balloon, flying the balloons, and what can be learned regarding solids of revolution.

Keywords: calculus, project, construction, solid of revolution, volume, surface area, hot-air balloon, 21st century skills

1. Introduction

Every year after the Advanced Placement (AP) Calculus exam, I find myself asking the same question: “How should I best spend my time with my Calculus students in the few remaining weeks of school?” As a general rule, AP students have been rigorously working to learn, prepare, and take their AP exam(s). By the end of the second week of May, even the most die-hard math enthusiasts in my classes are ready for a break from the strict regimen of homework and studying. After trying several different ideas to best utilize our time, I stumbled on a proposal from a friend and fellow math teacher. His suggestion was to apply my students’ understanding of calculus by having them utilize solids of revolution to design, analyze, build, and fly tissue-paper hot-air balloons.

2. The Design

Students were given the following prompt to set the stage for the last three weeks of the year:

Mr. Gasser is looking to start a side business of creating model hot-air balloons. Wanting to create the best balloon on the market, he has hired a number of firms to inform him what shape of balloon he should create to provide a product that stays in the air for the longest time possible and that flies higher than his competitors’ products. He has given each firm a set amount of material with which to work. The firm that submits the most convincing argument as to why their balloon is the best will be paid for their diligent efforts (to the tune of five bonus points each).

Students were broken into “firms” of two to four to tackle the problem and were given a list of the materials available to them. Each firm had access to the following materials:
• sixteen 20 × 30 inch sheets of art tissue paper (which when glued together end-to-end form eight 20 × 59 inch sheets of paper)
• a 36 × 70 inch sheet of bulletin board paper to use as a panel template
• a bottle of white glue and a glue stick
• approximately 30 inches of thin wire
• a few clothespins
• scrap cardboard to protect the desks and floors from glue
• a set of general directions for designing and building their balloons

The challenge for each firm was to work together as a team. This open-ended problem required them to think as mathematicians and utilize all eight of the Common Core Standards for Mathematical Practice (Common Core State Standards Initiative and others, 2010). In addition to learning how to construct a hot-air balloon from tissue paper, students were forced to take risks, collaborate, think critically, and communicate their results without direct instruction from me. These are all 21st century skills that each of our students should develop (Partnership for 21st Century Skills, 2009).

For the final assessment, students submitted a portfolio of their work. They included photos to document their design and construction process as well as their test flight. Documentation was supposed to be ordered logically and all work was to be typed. (This provided a great opportunity to introduce students to Microsoft’s Equation Editor). Below is a list of what I looked for in each portfolio along with the number of possible points that could be earned (for a total of 60 possible points):

- title page including group name along with names of all group members (2 pts)
- picture of balloon profile (2 pts)
- picture of panel template (2 pts)
- pictures (minimum three) of balloon construction (6 pts)
- piecewise defined function used to model profile curve (10 pts)
- detailed sketch of curves on a coordinate plane (5 pts)
- volume of balloon, including all supporting work (10 pts)
- surface area of balloon, including all supporting work (10 pts)
- explanation of process used to measure the maximum height of the balloon in its flight (5 pts)
- chart illustrating data collected (flight time, flight height, volume, surface area, basic profile design) (3 pts)
- revised profile designed after the first test flight, along with a convincing argument why the new, improved design should be chosen over all other designs (5 pts)

3. Constructing the Balloon

A majority of the time spent on the project was in designing and constructing the balloon. These steps required attention to detail and mathematical thinking. The students first designed a two-dimensional profile of their balloon. Then students constructed eight two-dimensional panels, or gores. Finally, each group fastened the panels together to make a three-dimensional balloon. Examples of some of the processes are shown below (see figures 2·3).

To better understand the design and construction process, it may be helpful to think of a beach ball (see figure 1). The profile of the ball is a circle. The corresponding panels are shaped like narrow diamonds with the middle vertices rounded out. When the panels are attached to each other along their seams they create a three-dimensional beach ball.

For each balloon, firms had to first design half of the profile of their theoretical balloon (see
Figure 1: A beach ball illustrating how, in this case, 6 panels sewed together approximate a sphere (public domain image, obtained from openclipart.org)

Figure 2: Sample profile design

Figure 3: Choosing a representative sample of points

This process gave rise to good discussion within the groups. Students discussed the size and shape of their balloon. They also discussed properties that they thought would make their balloon go higher than any other firm’s balloon. It was enjoyable to hear students critiquing the ideas and reasoning of their peers and to see them finally come together on one design that the firm could support. Each firm also had to make sure its balloon was small enough to be constructed with the given amount of art tissue paper, which led students to consider the relationship between the profile, panels, radius, and circumference. With some guidance, students found the arc length of their final profile curve could be no longer than the tissue paper (59 inches) and that their greatest circumference must be less than eight times the width of the paper (160 inches).

To ensure that their tissue paper would be long enough to form the balloon, firms needed to find the arc length of their balloons’ profiles. To do so, students laid a piece of string along the profile of the balloon, marked the endpoints of the profile on the string, and then measured the length of the taut string between the marks. Instructors could at this time teach students how to find the arc length of a curve if they would like their students to use their curve fitting abilities and calculus to calculate the arc length of the profile. Either way, students had to use mathematical thinking to answer the real-life, practical question, “Will we have enough tissue paper?”

After designing the profile, students then created a template for the eight panels, or gores. To save time I helped students through these steps, but if time permits, the task could be enriched by asking them to determine how to make the templates for the gores themselves. A brief outline of the concepts in designing a template follows. First, students were asked to draw their profile in the coordinate plane (see figure 2). Once drawn, the students identified a representative sample of points lying on the profile curve after first segmenting the curve into intervals of various widths (generally one to three inches), as shown in figure 3. When the panel template is constructed, the points along the profile curve will lie on the center axis of each panel. By finding the arc length (string method or calculus) from an end of the profile to a point on the profile, one can find a point on the center axis of the panel that corresponds to the point on the profile (see figure 4).

This process is repeated for each point on the profile. (This took some time as some students used over 100 points.) After finding each of the points on the center axis of the panel that correspond to the profile points, students needed to realize the width of the panel at each point is one-eighth of the circumference, since there will be eight equally sized panels creating the balloon.
The circumference can be calculated by using the radius from the matching point of the profile. I have found it easier to fold the panel in half, so the width of half of the panel would be one-sixteenth of the circumference. Repeating this process at each point gives the students a series of points that they can connect to create a template for the eight panels (see figure 5). When their edges are glued together, the eight panels will create the desired three-dimensional balloon.

Once students better understood the mathematics used to create the panels, students used their careful measurements of their profile to build a template for their panel. This was one of the more critical processes in the construction phase. Sloppy measurements or unfocused team members could lead to panels that, once glued together, create a balloon that looks nothing like what was expected.

The panel template (see figure 6) was made out of bulletin board paper. Once made, the students overlaid the template onto the pieces of art tissue paper to cut out the panels that formed their hot-air balloon. After the panels were cut out, students carefully glued them together, edge-to-edge, with a sufficient, yet minimal, amount of glue (either by using a glue stick or white glue) (see figures 7 and 8). The left edge of panel 1 was glued to the left edge of panel 2, the right edge of panel 2 was glued to the right edge of panel 3, and so on to form an accordion of panels. Then they curled back panels 2–7 and glued the right edge of panel 8 to the unglued edge of panel 1, completing the structure of the balloon. Once the panels were all glued together, a tissue paper cap was glued onto the top of the balloon to seal off the top seams of all of the panels. In addition, a thin piece of wire was used to provide rigidity and a small amount of weight to the bottom opening of the balloon (see figure 9).
4. Analysis of Design and Flight

After completing construction, and prior to flying their balloons, each firm analyzed its design. The design analysis helped students understand two underlying components of their balloon design: the volume and the surface area.

To explore these ideas more precisely, students used the coordinate points along the curve of their profile along with the regression capabilities of their graphing calculators or GeoGebra (Hohenwarter, 2014) to create a piecewise function that closely approximated the shape of their curve. They then used calculus to determine the volume and surface area of their balloon.

An example of a set of points from one profile (measured in inches), its corresponding piecewise defined function, graph, and calculations for surface area and volume are shown in figures 10 and 11, respectively.

Another goal of the project was for each firm to determine a method for measuring the total height achieved by its balloon as well as the heights of the balloons produced by the other firms. Several methods have been employed over the years to measure heights. Favorites have included using a clinometer (coupled with a little trigonometry) and range finders (supplied by the hunters in the class). Digital photographs from a fixed location have also been used so balloon flights can be compared side-by-side to gauge height. It was, however, very difficult to accurately measure the height of a balloon whose path was so easily changed by a slight breeze. In addition, an official time keeper was selected to measure the time each balloon stayed in flight.

5. Flying the Balloons

Needless to say, I was very cautious when it came time to introduce students to something that could burn them, their paper balloon, the school, or even the lush grass of the school baseball field. I was very aware of the fact that a little flame, some tissue paper, and a bit of dried glue could turn a pretty balloon into a frightful ball of fire, which would present more
Taking Calculus to New Heights

\[ f(x) = \begin{cases} 
5 & 0 \leq x \leq 5 \\
0.0848x^2 - 0.6509x + 6.0339 & 5 < x \leq 10 \\
-0.00006x^4 + 0.00548x^3 - 0.2119x^2 + 4.25843x - 18.20301 & 10 < x \leq 43 
\end{cases} \]

\begin{tabular}{|c|c|}
\hline
\textbf{x} & \textbf{y} \\
\hline
0 & 5 \\
1 & 5 \\
\vdots & \vdots \\
38 & 12.563 \\
39 & 11.125 \\
40 & 9.5625 \\
41 & 7.625 \\
42 & 4.375 \\
\hline
\end{tabular}

Figure 10: The scatterplot and corresponding piecewise defined function used to model the profile in figure 2 (created using GeoGebra)

\[ V = \pi \int_{a}^{b} r^2 \, dx \]
\[ = \pi \int_{0}^{5} 5^2 \, dx + \int_{5}^{10} (0.0848x^2 - 0.6509x + 6.0339)^2 \, dx + \int_{10}^{43} (-0.00006x^4 + 0.00548x^3 - 0.2119x^2 + 4.25843x - 18.20301)^2 \, dx \]
\[ \approx 24,733.779 \text{ cubic inches} \]

\[ SA = 2\pi \int_{a}^{b} f(x)\sqrt{1 + (f'(x))^2} \, dx \]
\[ = 2\pi \int_{0}^{5} 5\sqrt{1 + (0)^2} \, dx + \int_{5}^{10} (0.0848x^2 - 0.6509x + 6.0339) \sqrt{1 + (0.1696x - 0.6509)^2} \, dx + \int_{10}^{43} (-0.00006x^4 + 0.00548x^3 - 0.2119x^2 + 4.25843x - 18.20301) \sqrt{1 + (-0.000024x^3 + 0.01644x^2 - 0.4238x + 4.25843)^2} \, dx \]
\[ \approx 4,164.708 \text{ square inches} \]

Figure 11: Volume and surface area calculations
problems than I wanted. Having one or two fire extinguishers at hand along with the principal’s approval were prerequisites for flight day. The ultimate goal was to have students launch their balloons outside on a warm, still, spring day in May. This meant that I had to pay careful attention to the weather forecast, be patient, and take advantage of the nice weather when it came. After several experiments over the years, I have found that a propane turkey fryer provides the most consistent heat. To concentrate the heat, I placed some circular reducing funnels, supplied by a local heating and air conditioning company, on top of the turkey fryer to force the heat to exit a five inch diameter opening. Students were to have an opening no smaller than nine inches in diameter to leave space for their fingers so they could hold their balloon over the heat source (wearing gloves of course). See the cover image of this article for an illustration of the process.

Once the heat source was set up and producing a steady stream of heat, the students held their balloons over the heat source for up to five minutes each. Then they let go and anxiously hoped to see their balloon soar to the heavens. Some balloons have flown marvelously. Others did not fly so high but stayed in flight longer since they were designed with a large opening that would allow the balloon to act as a parachute on its descent. Some balloons leaked so much hot air that they only lifted a few inches. Team “Lead Balloon” illegally reinforced their seams with tape to prevent any air leakage. As a result of their construction, the balloon was so heavy that when released above the heat source it immediately sank down instead of floating up! Balloons heated with the turkey fryer have flown in excess of one hundred feet. Some have stayed in the air for over three minutes, while others ended their flight in a tree (at which time the clock officially stops).

6. Tying it all Together

Once all of the firms had an opportunity to fly their balloons, some even a second time, students returned to the classroom to share with the other groups. After sharing their flight times, heights achieved, profile design, volume, and surface area, firms then regrouped and created a new theoretical design. Each firm then prepared a written analysis about why the members felt their new design would produce a superior balloon.

If by this time in the project students have not yet fully discussed what keeps a hot-air balloon afloat, I like to facilitate a discussion on what exactly makes a hot-air balloon float. Students start to focus on how to maximize the amount of hot air in a balloon and how to keep it as hot as possible for as long as possible. Students soon realize that to keep the air hot in a balloon they need to minimize the surface area of the balloon to reduce the speed at which the hot air cools. I love seeing the light bulbs flick on when students realize that they already knew of a shape that maximizes volume while minimizing surface area—a sphere! Though we did not have time, it would have been interesting as a class to research why piloted hot-air balloons used in practice are not spherical.

7. Conclusion

Teaching seniors at the end of the year can sometimes be difficult, especially during those days when the sun actually shines in May. AP students know that the course is essentially over for them after they complete the AP exam. But in recent years, when I have utilized this balloon project, I have found my students to be thoroughly engaged. The remainder of the school year passed very quickly for all. While some balloons hardly flew at all and would have made a better wall decoration than an aircraft, some soared higher than I imagined possible. In the end, all groups learned firsthand through the invention process of success and failure. Calculus served as the tool to explain some of the “whys” behind both the successful and unsuccessful designs.

Throughout the project, students worked collaboratively with each other, took risks, had fun, and presented original solutions. These are all skills that will likely benefit them in the global marketplace that they will be entering in
a few short years. In addition, students had to utilize the Standards for Mathematical Practice (Common Core State Standards Initiative, 2014). These included making sense of a problem and persevering in solving it, constructing viable arguments and critiquing the reasoning of others, modeling with mathematics, using appropriate tools strategically, and attending to precision. Without question, students looked forward to class. Some even begged for more time to make adjustments after their first flights. Other students were even seen at lunch and in study hall discussing their designs. This past year multiple students commented on how this assignment was their favorite school project of all time. In future years, I will likely make adjustments to this activity, but for now, I have found a new favorite end-of-the-year calculus project, and I invite you to give it a try in your classroom.

Educators interested in more details on this particular project are encouraged to contact the authors.

References


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