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From the Editors…

Welcome to the Fall 2012 issue of the \textit{Illinois Mathematics Teacher}. It has been awhile, but we now have a great selection of articles to share with the members of ICTM.

This issue of the \textit{IMT} contains general interest articles as well as articles specific to a variety of levels of student learning. Deborah A. Crocker and Betty B. Long have written an article about using literature and technology to have students discover pi. Dianna Galante also addresses technology in her article about graphic organizers. Fractions are the topic of focus in an article by Geoffrey Lewis, Robert Hayes, and Marianne Wysocki. Two articles that may be of interest to high school teachers involve the mathematics associated with a playground swing and the mathematics of two famous curves. Rena Pate provides two articles in this issue that will be of interest to elementary level math teachers which discuss a readily available manipulative and problem solving. Charlotte Schulze-Hewett and Hana Sallouh discuss a manipulative that is readily available and how to implement it in your middle school or high school classroom. Finally, there is an article about using transformations in the high school geometry classroom.

We want to thank all the readers of the IMT. We have enjoyed being the editors of this journal for the past eleven years. As Marilyn retires, we are passing the job of editor to two other members of ICTM. We wish them the best of luck and hope you continue to enjoy the IMT under their direction.

Typically included at the end of each issue is a form for becoming a reviewer for the \textit{IMT}. This form has not been included in this issue due to page considerations. If you are interested in becoming a reviewer, please email imt@ictm.org and list the subject(s) or grade level(s) you would be interested in reviewing. In order to ensure that the articles are of interest to our readers, we send them to reviewers to get their approval. We would love to have reviewers willing to review in only one area or multiple areas. As postage prices continue to rise, we are trying to conduct most of our communications by email, so please update your information and your email address if you have not reviewed in the past year.

Please consider submitting an article or classroom activity to the \textit{IMT}. Consider writing about an activity that you use in your classroom and you would like to share with others. Your articles are needed to continue the sharing of ideas and the publishing of this journal on a regular basis.

Thank you for sharing.

\textit{Marilyn and Tammy}  editors
Using Literature to Get to Pi

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Literature is often used as a springboard into mathematics at the elementary and middle school levels. Stories can help motivate students to explore and discover mathematical concepts. Even if the reading level of the book is below grade level, the story can help students recall mathematical concepts and ideas. *Sir Cumference and the Dragon of Pi: A Math Adventure* is an example of this type of book. This activity incorporates literature in this manner and makes use of a handheld graphing device to discover pi using multiple representations. Some prior knowledge of a handheld graphing device is assumed.

As you read the story to your class, they will be introduced to the characters Sir Cumference, his son Radius, and his wife Lady Di of Ameter. During lunch one day, Sir Cumference is accidentally turned into a dragon when Radius brings him a bottle labeled “Fire Belly” from the doctor’s workroom to cure his stomachache. In the doctor’s workroom, Radius looks for a cure to change his father back. He finds a container with a poem written on the outside that says:

Measure the middle and circle around,
Divide so a number can be found.
Every circle, great and small –
The number is the same for all.
It’s also the dose, so be clever,
Or a dragon he will stay . . .
forever. (p. 13)

Radius realizes he must solve the riddle in the poem involving a circle in order to change his dad back from a dragon into a man. At this point, Radius goes to the carpenters, Geo of Metry and his brother Sym. While there, Radius has an idea about “across the middle” and “around the circle” (p. 15). For the next several pages, in the story, Radius measures various circular items. In table 1 is a list of the items that Radius finds and their measures.

<table>
<thead>
<tr>
<th>Item</th>
<th>Across the Middle (inches)</th>
<th>Around the Outside Edge (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wheel</td>
<td>49</td>
<td>154</td>
</tr>
<tr>
<td>Onion Slice</td>
<td>3.5</td>
<td>11</td>
</tr>
<tr>
<td>Basket</td>
<td>7</td>
<td>22</td>
</tr>
<tr>
<td>Bowl</td>
<td>14</td>
<td>44</td>
</tr>
<tr>
<td>Round of Cheese</td>
<td>17.5</td>
<td>55</td>
</tr>
</tbody>
</table>

Students should measure other circular objects such as tops of soda cans, hula-hoops, cookies, or plastic lids and add their measurements to table 1. The data can be entered on any handheld graphing device with the capability for scatterplots and linear regression models. In this article, we will show screens captured from the TI-73. The final table will be entered into the list editor on the TI-73. The “Across the Middle” measurements are in L1 and the “Around the Outside Edge” measurements from the literature book are in L2 (see fig. 1).

![Fig. 1 Screen shot of L1 and L2](image-url)
The data in the list editor can be analyzed both graphically and numerically. Numerically, you can have the students create lists L3, L4, L5, and L6 containing \( L2/L1, L2 + L1, L2 - L1, \) and \( L2* L1, \) respectively. By analyzing the sum, difference, product and quotient, students should observe that only the quotient appears to be approximately constant. (see fig. 2).

Fig. 2 Lists showing ratio of “Around the Outside Edge” to “Across the Middle”

Ask the students if they see anything that the numbers in L3 have in common. They should notice that all of the numbers in the new list are a little more than three. A good representation for all of the numbers in L3 might be the mean. Go back to the home screen by pressing \( 2^{\text{nd}} \text{ MODE} \) (for QUIT) and press \( 2^{\text{nd}} \text{ STAT MATH Mean( )}. \) Be sure to indicate, by typing, that you want the mean of L3 and press \( \text{ENTER} \) (see fig. 3).

Fig. 3 Screen shots of commands for finding the mean

The students should notice that the mean of the numbers in L3 is approximately 3.14. To analyze the data graphically, set up a stat plot (see fig. 4).

Fig. 4 Screen shots showing how to set up a stat plot

The data in the scatter plot should appear linear as in figure 5.

Fig. 5 Scatter plot of data from the story

Have the students estimate the slope of the line and the y-intercept using points on the stat plot and the mathematics they have studied. Then ask them to choose manual fit from the stat menu. Students can position the cursor at a starting place (see + sign on the second screen in figure 6), press \( \text{ENTER} \) and use the arrow keys to create a line segment they believe is a good model for the data.

Fig. 6 Screen shots of Manual-Fit Options

When they have the segment they want, they press \( \text{ENTER} \) again to see the equation of their line at the top of the screen (see fig. 7).

Fig. 7 Manual-Fit line with equation
Students should start with the segment they want and adjust the slope by pressing the up or down arrow keys on the handheld graphing device and adjust the y-intercept by pressing the left or right arrow keys on the handheld graphing device, as needed, to find a model for the linear data (see fig. 8). The model will be displayed at the top of the screen and can be stored as Y1.

What do you notice about the slope? It should be close to the numerical average you found in the list L3 earlier. Let's compare the students’ model to the line of best fit generated by the handheld graphing device. Press STAT CALC LinReg (see fig. 9).

Like the graph shown in figure 10, students can look at a graph of their model and the line of best fit generated by the handheld graphing device.

Fig. 10 Graph of the line of best fit and the model found using Manual-Fit

After this numerical and graphical analysis of the data from across and around several circles, the students should draw the conclusion that the dosage needed to change Radius’ father back into a man is approximately 3.14 spoonfuls. This is the approximate ratio of “around” to “across” in any circle. In the story, Radius saved his father by computing the correct dosage of the potion to give him. That dosage turned out to be pi spoonfuls. We have approximated pi numerically and graphically and understand that it is a ratio that holds true for any size circle. To celebrate, Sir Cumference, Radius, and Lady Di of Ameter joined all the people of the kingdom for a piece of pie.

NOTE: The teacher could give all of the students a cookie or something round as a treat after this activity.

References

Using the SmartArt™ Organization Chart to Create a Graphic Organizer

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One way to increase understanding and develop problem solving skills in your mathematics class is to develop lessons that appeal to the visual learner. You can accomplish this by inserting graphic organizers when preparing worksheets and cooperative learning activities. A graphic organizer can help your students tackle difficult concepts by providing a visual tool to help arrange key ideas and definitions, organize categories, or analyze mathematical processes. By helping students present what they know visually, you can increase student interest and motivation while promoting conceptual understanding in your classroom.

You can use an organizer to aid students with understanding the sequence in a process or to group related information. Using graphic organizers can also help students with developing reading strategies in mathematics by supporting their ability to break apart difficult text, to group related information, or by providing a structure for what they already know.

Supporting Problem Solving

This ability to create and use a visual representation of a problem supports and enhances students’ problem-solving strategies. According to the Principles and Standards for School Mathematics (PSSM) [National Council of Teachers of Mathematics (NCTM), 2000], “different representations support different ways of thinking about and manipulating mathematical objects” (p. 361). Using a variety of representations in instruction should enable students to:

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representation to model and interpret physical, social and mathematical phenomena (p. 360)

One limitation with adding a graphic organizer to your lesson plan is the amount of time and effort needed to create a high-quality document. One easy option available with Microsoft Word™ is the use of the SmartArt graphic tool on the Insert tab. You can quickly create a graphic organizer and drop it into your document like a piece of clipart, then easily copy and paste it, move it, or resize it. One graphics template useful for mathematics is the Organization Chart which can provide a picture of a hierarchical relationship or create a decision tree. Another useful template is the Basic Venn diagram which can be used to visually display the union and intersection of sets.

Getting Started

To get started, open a new document in Microsoft Word and then select the Insert tab and SmartArt. Click on the SmartArt button (Figure 1) near the center of the toolbar. Your toolbar may appear differently depending on the software version you are currently using. The Diagram Gallery dialog box with a pallet of diagrams will appear (Figure 2). There are six diagram types available including List, Process, Cycle, Hierarchy, Relationship, Matrix and...
Pyramid. Click on the diagram type you want to select and then click OK. As an example, select Hierarchy, then Organization. The diagram template will appear with guides (Figure 3). At this point you can type text, add additional graphics, or change the layout of the diagram depending on the template you selected. Finally, resize or move the graphic to fit your document. You can experiment with the many options and then use the UNDO button to backtrack to the best choice.
Classroom Uses

Example 1: In a geometry class, you can use the Organization Chart to help students describe the properties of quadrilaterals. Individually or in groups, ask students to describe the properties of each type of quadrilateral by supplying information about sides, angles and diagonals in the chart (Figure 4).

Example 2: In an algebra class, the Organization Chart can chart help students focus on an important property of the slope of a line. From a list of equations, have students first put each equation in slope-intercept form. Next, ask students to identify the slope value and place the equation in the appropriate slot on the chart (Figure 5). Put each equation in slope-intercept form then complete the chart:

1. 5.
2. 6.
3. 7.
4. 8.

![Figure 4 Properties of Quadrilaterals](image)

![Figure 5 Solution for Example 2](image)
Create a Decision Tree

Example 3: You can use the Organization Chart in a probability and statistics class to support a problem-solving strategy that uses a decision tree to help clarify a difficult topic. The accompanying tree diagram (Figure 7) represents a two-stage experiment where students use the “branches” of the tree to help calculate probabilities.

Vector Electronics – The circuit boards for new laptop computers are manufactured at three different locations and then shipped to the main plant of Vector Electronics for final assembly. Plants A, B, and C supply 50%, 30% and 20%, respectively, of the circuit boards used by Vector Electronics. Production has been carefully monitored by the quality control department. It has been determined that 1% of the circuit boards produced by Plant A are defective, whereas 2% of the circuit boards for Plants B and C are defective. If a Vector Electronics computer is selected at random, and the circuit board is found to be defective, what is the probability that the circuit board was manufactured in Plant C? Use the tree diagram to help with the calculation of .

Solution: Let A, B, and C denote the events that the laptop computer selected has a circuit board from Plant A, B, or C, respectively. Let D indicate the event that the laptop has a defective circuit board. The complement, \( D^c \) indicates no defect. Construct a tree diagram (Figure 7) to represent the situation and supply values for the equation below.

\[
\begin{align*}
\text{Find the probability} \\
\text{A} & \quad .50 \\
\text{B} & \quad .30 \\
\text{C} & \quad .20 \\
D & \quad \text{.01} \\
D^c & \quad .99 \\
D & \quad .02 \\
D^c & \quad .98 \\
D & \quad .02 \\
D^c & \quad .98
\end{align*}
\]

Example 4: You can use a graphic organizer to help students select a mathematical representation that best suits their learning style or to help translate between different representations. The Basic Venn graphic organizer can be found by clicking on the SmartArt icon then selecting the Relationship group then Venn.

Senior Class – At Midwest High School there are 150 students in the senior class. A total of 30 students are enrolled in Physics II, 35 students are enrolled in A. P. Calculus, and 100 students are enrolled in English IV. There are 15 students in physics and calculus, 15 in physics and English, and 20 in calculus and English. Five seniors signed up for all three classes. How many seniors are enrolled in the three classes? How many seniors are not taking any of the three classes?

Solution 1: Using a symbolic representation, students would use a counting formula for three sets where

\[
\begin{align*}
\text{number of students taking physics} \\
\text{number of students taking calculus} \\
\text{number of students taking English}
\end{align*}
\]

Then
\[ n(P) + n(C) + n(E) - n(P \cap C) - n(E \cap C) - n(P \cap E) + n(P \cap C \cap E) = 30 + 35 + 100 - 15 - 15 - 20 + 5 = 120 \]

150 – 120 = 30 the number of students not in any of the classes

Solution 2: Using a graphic organizer as a visual aid, students label each section of the Basic Venn diagram (Figure 8) for the three overlapping sets.

Helpful Information and Available Support

The Microsoft website offers a multitude of support options for use with SmartArt including additional graphics, tips, sample ideas, and online tutorials. To visit the website, go to http://office.microsoft.com and select the Help and How-to tab.

References


Plan to attend the NCTM Regional Conference at the Hyatt Regency Hotel in Chicago on November 28-30, 2012.
One of the most difficult topics for middle school students is fractions. Students spend long hours on worksheets that have them practicing the rules for addition, subtraction, multiplication, and division of fractions. Yet when time comes for testing teachers are at a loss to explain why their students perform poorly. The same can be said to a lesser extent for the student scores on decimals and percent.

The National Council of Teachers of Mathematics addresses the teaching of these topics in the Standards for Grades 6-8 (Principles and Standards for School Mathematics 2000). The specific goals are:

- That students be able to work flexibly with fractions, decimals and percents.
- Select appropriate methods and tools for computing with fractions, decimals and percents.
- Understand the meaning and effects of arithmetic operations with fractions, decimals and integers.
- Select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil.
- Develop and analyze algorithms for computing with fractions and decimals.
- Develop and use strategies to estimate the results of computation and judge the reasonableness of the results.

Is this what is happening in our middle schools? No! What we see almost without exception is a series of worksheet after worksheet that seem to provide no increase in standardized test scores and high school students that continually answer \( \frac{1}{2} + \frac{1}{4} = \frac{2}{6} \).

When you ask any high school teacher if their students can do rational number operations in algebra or geometry the answer is almost always no. What happened to all the time spent on worksheets? Was there any learning? Are we just teaching our students what key strokes to perform on a calculator and hoping they can even remember this? Read the assigned chapter, answer the questions at the end of chapter, sit and take lecture notes, complete the worksheet, and take a test are all listed as this is not teaching in How to be an Effective Teacher, (by Harry Wong, pp. 218-219, 1998). Bloom’s taxonomy arranges the verbs into six related groups:

1. Knowledge,
2. Comprehension,
3. Application,
4. Analysis,
5. Synthesis,

Most worksheets involve answering only level 1 type questions. NCTM standards start at level 2 type questions. No wonder children have trouble remembering or learning fractions. NCTM standards talk about communication and connections. Each is very difficult to ask, evaluate, or teach with traditional worksheets.

The solution for more effective lessons involves both the NCTM standards and being an effective teacher. Elementary
teachers have long known that they must establish connections for their students to understand fractions. Often these connections involve making a drawing or diagram of the concept or problem. The effective teacher is the one who has the students doing the work. The one doing the work is the one doing the learning. Often times middle school teachers end up with a high degree of frustration and are worn out because they are the ones doing the work and their students are not doing the learning.

Middle school teachers must spend more time having their students working on developing concepts and less time on abstract practice. This can be accomplished by mind mapping fractions, decimals and percents as an interrelated activity with a connection to the kinesthetic activity of making the related picture or model. Students must see the marriage of picture, fraction, decimal, and percent. Once the basics are well understood it is easy to progress to more difficult problems.

The mind map consists of 5 separate but related parts. A rectangle to show fractions, a dollar sign to show the value in money, a decimal point to show the decimal value, a triangle to show the percent and a 10 by 10 grid to show the appropriate picture.

The first part: rectangle, $, decimal, %, grid are easy for the students to remember. This forms the first part of the connection. The second part of the connection is the pictorial representation of the fraction, decimal and percent. This connected to money, something that all students see and think about each day provide a strong memory basis for the students. The 10 by 10 grid is the picture method recommended and used by the NCTM in the standards.

It is not necessary to use large or complicated fractions. Only the most common should be used, halves, thirds, fourths, fifths, eighths, and tenths. These provide more than an adequate background for a fundamental understanding of fractions decimals and percent. If the students have a firm understanding of the basics of fractions the rest is easy.

A few examples are shown, put in order so that students can also see the equivalent values:

\[
\begin{align*}
\frac{1}{2} & \quad \$ .50 \quad 0 \cdot .50 \quad 50\% \\
\frac{1}{3} & \quad \$ .33 \quad 0 \cdot .33 \quad 33\% \\
\frac{2}{3} & \quad \$ .66 \quad 0 \cdot .66 \quad 66\% \\
\frac{1}{4} & \quad \$ .25 \quad 0 \cdot .25 \quad 25\% \\
\frac{2}{4} & \quad \$ .50 \quad 0 \cdot .50 \quad 50\% \\
\frac{3}{4} & \quad \$ .75 \quad 0 \cdot .75 \quad 75\% \\
\frac{4}{4} & \quad \$ 1.00 \quad 1 \cdot .00 \quad 100\% 
\end{align*}
\]
Use all the fractions to help with the connections. You need to use all the halves, all the thirds, all the fourths, all the fifths, all the eighths, and all the tenths. Only a sample is shown above. It is necessary to show all of these to show the connections.

It is now easy for students to find equivalent fractions. They are the ones with the pictures that have shaded the same. The comparison of order of fractions decimal and percent is also as simple as looking at the picture. Basic fractions are easy to reduce simply by using the illustration. Decimal and percent equivalents to fractions can be very visual. Performing addition and subtraction with and without a common denominator is as easy to do as counting squares. More importantly pictures aid students in making estimates. This is especially important if the students are using a calculator. Is this the right answer? Or have I made a mistake with the decimal or denominator. The use of money, with the picture also reinforces the concept of estimation. Estimation is one of the key goals of the NCTM. Multiplication and division of fractions and decimals makes sense to students in picture and money form. No more ½ of $.50 is 1.00! Large posters of the basics should be placed around the room for easy and constant reference. Once the basic fractions and operations have been mastered most students can easily make the transformation to more complicated problems. Not that the problems are any more complicated only that the numbers are larger. Is this important for your students to know? Do we really add sevenths and eights or multiply $2\frac{3}{7} \times 4\frac{1}{3}$. Does this really have a real life connection? Is it tested on State exams or the ACT test? Use your time to teach what they will need in their real life.

Note the physical mind mapping connections: There is the brain friendly connection involving the pictures and symbols. Fractions are shown with a rectangle so that the fraction bar fits parallel to the top and bottom of the rectangle. The decimal point follows the decimal with the dollar sign. A triangle is used to show percent because the side of the percent sign resembles the side of a fraction. Squares to show pictures are drawn in a $10 \times 10$ format to match money, percent and NCTM standards. (NCTM, p. 215) The visual connections must be the first established for the students to make the connections between fractions, decimals and percent. Other connections are shown in money and picture scale. Yes a few worksheets are necessary for practice and review. Yes a few larger problems are necessary, but this is a question of balance. Now the scale has slipped to more and more problems with less and less understanding of the marriage of the basic concepts that bring fractions, decimals and percent together. Calculators can perform the most difficult computation problems, but students must be able to interpret the correctness of their results and they must also be able to make a connection between the use of the calculator and the basics problems. Fractions, decimals and percent should not be taught as individual activities, but must be taught as different representations of the same processes for students to have complete understanding.

When students see these connections, they can choose the method of problem solving that makes most sense to them as recommended by the NCTM standards. Finding 50% of an unknown number is the same as finding $\frac{1}{2}$ of it or in decimal form multiplying by .50. The picture provides the final check of estimation. Have they shaded in the appropriate amount of the given? Likewise, why does dividing by $\frac{1}{2}$ make the number larger? Try dividing by .50 and see what happens? This method also appeals to a variety of learning styles, most importantly
the visual and kinesthetic learners’ needs are met.

This all fits with current research that says explaining basic concepts behind math problems improves children’s learning. “New research from Vanderbilt University has found students benefit more from being taught the concepts behind math problems rather than the exact procedures to solve the problems. The finding offers teachers new insights on how best to shape math instruction to have the greatest impact on student learning. The research by Bethany Rittle-Johnson, assistant professor of psychology and human development at Vanderbilt University’s Peabody College and Percival Mathews, a Peabody doctoral candidate, is in press at the Journal of Experimental Child Psychology. Teaching children the basic concepts behind the math problems was more useful than teaching children a procedure for solving the problems—these children gave better explanation and learned more. …This adds to the growing body of research illustrating the importance of teaching children concepts as well as having them practice solving problems.” (Science Daily, April 2009). The shapes, the relationships and the pictures all add to the student’s understanding of the concepts of fractions, decimals and percents.

The internet can also provide a valuable tool for both the practice and the understanding of fractions, decimals and percents. There are many web sites that can provide a wide variety of interactive activities and practice that would not be possible for the classroom teacher to duplicate. This provides the help in understanding that is fun and involved for today’s learners. Working math now will become a fun activity and not a bore. These sites must be carefully chosen to be of interest to the students and sound mathematical content.

Look how these activities fit together with the NCTM standards. Students are working flexibly, they understand the meaning, they can select from the appropriate method or tool, they can analyze the different algorithms, they can develop strategies, and estimate the reasonableness of the results. Always refer back to the posters. Students will need to see the connections many times before it all makes sense to them.

New state and national testing standards relate to Bloom’s taxonomy. Level one type questions, knowledge, are rarely asked. Students can simply crunch it out on a calculator. Comprehension, applications, analysis, synthesis, and evaluation are the types of questions asked. This requires a different approach to the teaching of mathematics than the worksheets of the past. Students must evaluate, develop, compare and contrast different methods of doing the problems in order to fully understand the relationships of fractions, decimals and percents.

Years of teaching experience and research now offer all teachers methods of providing effective instruction. State testing evaluates how successful we are at teaching standards using research based methods. This article provides a start in the directing of successful research based instruction.
Abstract

In this paper, the authors analyze the math and science in a common piece of playground equipment, a swing. The authors derive the equations needed to describe the motion of a swing. Next, they perform an actual experiment with a real swing and then give teachers some suggestions on how to use this activity in their classroom.

In every community, you can find a children’s playground and Forest Park, IL is no exception. This playground is on the corner of Randolph and Circle in Forest Park and contains a lot of different equipment. One might think that this playground equipment for children is very simple but actually there is a lot of mathematics and physics used in the development of this equipment.

The playground has several different pieces of equipment, but the focus for this paper is on the swing set. The mathematics and physics involved in the swings uses similar triangles, trigonometry, Newton’s Second Law and much more.

The diagrams and equations below are idealized in order to simplify the mathematics. In a real experiment using a swing, there would be several different possibilities, which would greatly complicate the mathematics and/or the measurements. Friction introduces a force other than gravity and if we do not assume a straight line, then we cannot use similar triangles in our explanations. Here are some of the assumptions being made: 1) there is no air friction, 2) the swing swings in a straight line not in an arc, 3) the weight of the chain is ignored and 4) the length of the swing is the distance from the pivot point to the center of mass.

In the diagram below, l is the length of the chain, P is the point where it is attached, E is the equilibrium position and A is the maximum position. Pull the swing out to A. The swing is the projection on the X-axis of a point moving on a circle.

Figure 1
\[ x = A \cos \left( \frac{2\pi t}{T} \right) \quad y = A \sin \left( \frac{2\pi t}{T} \right) \]

\[ v = -v_{\text{max}} \sin \left( \frac{2\pi t}{T} \right) \quad v_{\text{max}} = \frac{2\pi A}{T} \]

\[ a = -a_{\text{max}} \cos \left( \frac{2\pi t}{T} \right) \quad a_{\text{max}} = \frac{4\pi^2 A}{T^2} \]

Where

- \( T \) is the period or the time for one vibration (or rotation)
- \( x \) is the displacement from the equilibrium position (E)
- \( t \) is the time from when the swing is released
- \( A \) is the amplitude or how far the swing is pulled back before it is released
- \( v \) is the speed of the swing at time \( t \)
- \( v_{\text{max}} \) is the highest speed of the swing (at \( x = 0 \))
- \( a \) is the acceleration of the swing
- \( a_{\text{max}} \) is the highest acceleration of the swing (at \( x = A \))

The force equation for the swing can be found from similar triangles.

Then the sum of these forces equals the total force \( (F) \) on the swinger.

Assuming the swing is not pulled back very far, the force triangle is similar to the triangle formed by the swing where \( (l) \) is the distance from the swinger to the pivot and \( (X) \) is the distance the swing is pulled back.

The triangles are similar and therefore:

First draw the forces acting on the swinger. They are gravity \( (mg) \) pulling down and the rope \( (\tau) \) pulling up along the rope.
Where \( x \) is the displacement from the equilibrium position (\( x = 0 \)). The maximum position (\( A \)) is how far the swing is pulled back and \( l \) is the length of the swing from the equilibrium position to the center of mass. The figures are force diagrams of the forces on the swing. If \( x \) is positive, then \( F \) is negative as are \( v \) and \( a \). \( A \) is the amplitude or maximum value of \( x \). It “comes into play” in the general equations for position, velocity and acceleration.

Newton’s Second Law is \( F = ma \), so
\[
-\frac{ma}{x} = \frac{mg}{l} \Rightarrow -ma = \frac{mgx}{l}
\]
\[
\Rightarrow -a = \frac{gx}{l}.
\]
(Note: acceleration is defined as \( \frac{d^2x}{dt^2} \))
\[
\Rightarrow -\frac{d^2x}{dt^2} \cdot \frac{gx}{l} = 0,
\]
then assuming a trigonometric solution, we get
\[
x = A \cos \frac{2\pi t}{T}
\]
\[
\frac{dx}{dt} = -\frac{2\pi}{T} A \sin \frac{2\pi t}{T}
\]
\[
\frac{d^2x}{dt^2} = -\frac{4\pi^2}{T^2} A \cos \frac{2\pi t}{T}
\]
and by substitution in
\[
-\frac{d^2x}{dt^2} \cdot A \cos \frac{2\pi t}{T} = 0
\]
we get
\[
\frac{4\pi^2}{T^2} A \cos \frac{2\pi t}{T} = -\frac{g}{l} \frac{A \cos \frac{2\pi t}{T}}{l} = 0
\]
\[
\frac{4\pi^2}{T^2} = \frac{g}{l}
\]
\[
T = 2\pi \sqrt{\frac{l}{g}}
\]
which is the time for one swing.
\[
x = A \cos \frac{2\pi t}{T}
\]
\[
\frac{dx}{dt} = -\frac{2\pi}{T} A \sin \frac{2\pi t}{T}
\]
\[
\frac{d^2x}{dt^2} = -\frac{4\pi^2}{T^2} A \cos \frac{2\pi t}{T}
\]

<table>
<thead>
<tr>
<th>( t )</th>
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<tbody>
<tr>
<td>0</td>
<td>( A )</td>
<td>0</td>
<td>( -A \cdot \frac{g}{l} )</td>
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<tr>
<td>0.25T</td>
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<td>0.50T</td>
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<td>( A \cdot \frac{g}{l} )</td>
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<tr>
<td>0.75T</td>
<td>0</td>
<td>( A \cdot \frac{g}{l} )</td>
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<td>( T )</td>
<td>( A )</td>
<td>0</td>
<td>( A \cdot \frac{g}{l} )</td>
</tr>
</tbody>
</table>

Note to the teacher. The table serves a summary of all of the possibilities. Please adapt this activity to suit your class. For your class, you may want to use only the time equation and use a pendulum or swing to check it. Or, perhaps you want your class to work with variables, such as, changing
the length of the swing, weight, etc. You could also take your class to the playground and have the students take turns swinging on the swing and timing the person on the swing and recording this information. If it is not possible to do this activity during class, it could be given as a homework assignment or students could hang weights from their desks. When this is done the student changes the length of the string and observes the change in period ($T$). In this paper, the length of the chain is constant.

The next section of this paper, describes what happened when we went to the actual swing set and tried out our theories.

When the equation for the period ($T$) of the swing was derived, several assumptions were made. Going to a playground and measuring $l$ and $T$ can test these assumptions. The equation was

$$T = 2\pi \sqrt{\frac{l}{g}}$$

where $T$ is the period or the time for one swing (back and forth), $l$ is the length of the swing from the pivot point at the top to the seat at the bottom, and $g$ is the gravity constant 9.8 m/s$^2$.

Procedure:

1. Measure $l$
2. Predict $T$ using the equation
3. Measure $T$ by timing 10 complete swings and dividing by 10

**Trial 1** – with no person on the swing

1. $l$ is about 1.89m
2. $T = 2\pi \sqrt{\frac{1.89}{9.8}}$ and, therefore
   $$T = 2.76\text{ sec}.$$ 
3. Using a stop watch time ten swings of the swing, which is 2.43sec.

We found our percent of error to be 14%, which was far too much. This calculation was made by assuming that 2.43 was the correct answer. The % of error is $\frac{2.76-2.43}{2.43} = 0.1358$ or 13.58% or 14%.

We decided that the error lay in the length of the swing. Also, when we derived the equation we placed the weight vector “$mg$” (see fig. 2) at the bottom of the swing where we assumed the weight was concentrated. However looking at the actual swing, it is clear that most of the weight is in the chains and not in the seat. So the force vector should be placed further up the chain.

If we substitute 2.43 for $T$ in the equation, we find the effective $l$ to be 1.47m, quite different from our measured distance 1.89m.

In the next trial, we sat on the swing and our weight of 170 pounds became the most significant weight in the system. This time we measured $l$ from the pivot point at the top to where we estimated the center of mass of the person on the swing was.

**Trial 2** – with a person on the swing

1. $l$ is about 1.6m
2. $T = 2\pi \sqrt{\frac{1.6}{9.8}}$ and $T = 2.54\text{ sec}$
3. The time for ten swings is 2.66sec

We next found the % of error $\frac{2.66-2.54}{2.54} = 4.7\%$ which is more reasonable.

We recalculated $l$ this time and got 1.75m and we guessed that we were off by 15cm in estimating the center of mass of the rider.

Another assumption that was made in deriving the equation for $T$ was that the swing moves in a straight line ($x$ in fig. 2). We know the swing actually curves but we thought that if we kept the amplitude
(amount of swing) small, we could assume that the difference between the arc of the circle the swing actually makes and the length of the chord of that arc would be small. If we use amplitude of 0.3m and the swing length was 1.89m, then the angle is \( \frac{0.3}{1.89} \) radians or 9 degrees. 

We tested this assumption by swinging as high as possible.

**Trial 3** – with a person on the swing and high swing

1. \( l \) is 1.7m
2. \( T = 2.62 \) sec
3. Time ten swings at 2.83sec, which result in an 8% error.

This time we estimated our amplitude to be 1.2m so our angle is \( \frac{1.2}{1.89} = 0.635 \) radians or 36 degrees. Our conclusion was that keeping the angle under 10 or 15 degrees would give good results but big swings could throw the answer off.

This activity could be used in any grade from 3rd to calculus. Some teachers may need to talk with their science colleagues for some extra guidance. Regardless, even at the earliest grades, the students could measure “\( l \)” and calculate \( T \) from the equation \( T = 2\pi \sqrt{\frac{l}{g}} \) or \( T^2 = 4\pi^2 \frac{l}{g} \). Then they could measure \( T \) (and find \( T^2 \)) and compare the answers. Eighth graders could calculate the percent of uncertainties and discuss the effects of air friction and the straight-line assumption. The students could calculate \( x, v, \) and \( a \) from the motion results. And calculus students could integrate the acceleration equation to find the velocity at various points. This activity can generate a lot of discussion and questions and definitely show how math and physics is applied even to children’s playground equipment. This should settle the questions of “when am I (or anyone) ever going to use math?”

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Check out the ICTM website at ictm.org. The latest issue of the ICTM Bulletin is available here as well as information about the officers and directors. You may want to share the scholarship information with your students. Consider nominating a colleague for one of the ICTM awards. All of this and more is on the website – check it out.
Variations on the Witch of Agnesi

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Introduction

At some point in their mathematical training, most teachers encounter the Witch of Agnesi. The Witch, a plane curve named and studied in the mid-1700s by the Italian mathematician Maria Agnesi, is constructed as follows:

1. Center a circle with radius $a$ at the point $(0,a)$.
2. Choose any point $A$ on the line $y=2a$ and connect it to the origin $O$ with a line segment.
3. Label the point where the line segment $OA$ intersects the circle as $B$.
4. Let $P$ be the point whose $x$-coordinate is the same as point $A$ and whose $y$-coordinate is the same as point $B$.

The Witch is the collection of all points $P$ obtained by moving the point $A$ along the line $y=2a$. (Figure 1)

If we use $\theta$, the angle that the line segment $\overline{OA}$ makes with the $x$-axis, as a parameter, the equation of the line $\overline{OA}$ is seen to be

$$y = \begin{cases} x \tan \theta, & \theta \neq \frac{\pi}{2} \\ 2a, & \theta = \frac{\pi}{2}. \end{cases}$$

We get a piecewise representation since the tangent of a right angle is undefined. The equation of the circle on which the construction is based is also easily determined, since we know both its center $(0,a)$ and radius: $x^2 + (y-a)^2 = a^2$.

To find the coordinates of any point $P$ that lies on the Witch:

1. We find the $x$-coordinate of the point $A$, the point where the line $\overline{OA}$ intersects the line $y=2a$. Using the parameterization we constructed above, if $\theta \neq \frac{\pi}{2}$, we have $x \tan \theta = 2a$, or $x = 2a \cot \theta$. If $\theta = \frac{\pi}{2}$ the line $\overline{OA}$ is precisely the $y$-axis, and so $x = 0$.
2. We then find the $y$-coordinate of the point $B$, the point where $\overline{OA}$ intersects the circle. If $\theta \neq \frac{\pi}{2}$, we determine where the line $y = x \tan \theta$ intersects the circle $x^2 + (y-a)^2 = a^2$.
\[ x^2 + (y-a)^2 = a^2 \]
\[ (y \cot \theta)^2 + (y-a)^2 = a^2 \]
\[ y^2 \left(1 + \cot^2 \theta\right) - 2ay = 0 \]
\[ y^2 \left(\csc^2 \theta\right) - 2ay = 0 \]
\[ y \left(y \csc^2 \theta - 2a\right) = 0 \]

Since our geometric understanding of the problem tells us that \( y \neq 0 \), we must have

\[ y \csc^2 \theta - 2a = 0 \]
\[ y = \frac{2a}{\csc^2 \theta} \]
\[ y = 2a \sin^2 \theta \]
\[ y = a \left(1 - \cos 2\theta\right) \]

using the double angle identity \( 2 \sin^2 \theta = 1 - \cos 2\theta \). Note that if \( \theta = \frac{\pi}{2} \), no computation is necessary. We have \( y = 2a \).

This allows us to describe the Witch parametrically using the following equations:

\[ x = 2a \cot \theta \]
\[ y = a \left(1 - \cos 2\theta\right) \]

where \( 0 < \theta < \pi \). Now that we have the curve described parametrically, we can eliminate the parameter \( \theta \) to describe the Witch explicitly as a function of the variable \( x \).

\[ y = a \left(1 - \cos 2\theta\right) \]
\[ y = \frac{2a}{\csc^2 \theta} \]
\[ y = \frac{2a}{1 + \cot^2 \theta} \]

\[ y = \frac{2a}{1 + \left(\frac{x}{2a}\right)^2} \]
\[ y = \frac{8a^3}{x^2 + 4a^2} \]

The function described by this equation is continuous, with a graph that is symmetric about the \( y \)-axis and asymptotic to the \( x \)-axis. Some simple calculus reveals the surprising fact that while the Witch lies entirely above the \( x \)-axis, the area of the region bounded by the Witch and the \( x \)-axis is a finite value:

\[ \int_{-\infty}^{\infty} 8a^3 \, dx = 2a \int_{-\infty}^{\infty} \frac{dx}{1 + \left(\frac{x}{2a}\right)^2}. \]

Make the change of variables \( u = \frac{x}{2a} \):

\[ 2a \int_{-\infty}^{\infty} \frac{dx}{1 + \left(\frac{x}{2a}\right)^2} = 4a^2 \int_{-\infty}^{\infty} \frac{du}{1 + \left(\frac{u}{2}\right)^2} = 4a^2 \tan^{-1} u \bigg|_{-\infty}^{\infty} = 4\pi a^2. \]

For those teachers whose students enjoy experimenting with this type of construction, a follow-on exercise can easily be created by replacing the circle used to draw the Witch with another geometric shape, such as a square or a triangle. Two resulting curves, which our students have named “Maria’s Bridge” and “Agnesi’s Tent,” while perhaps not as elegant as the original Witch, are interesting to construct and study in their own right.
Maria’s Bridge

To construct Maria’s Bridge, we replace the circle used to construct the Witch of Agnesi with a square.

1. Position a square with side length $2a$ so that the midpoint of its lower edge is at the origin $O$.
2. Choose any point $A$ on the line $y = 2a$ and connect it to the origin $O$ with a line segment.
3. Label the point where $OA$ intersects the square as $B$.
4. Let $P$ be the point whose $x$-coordinate is the same as point $A$ and whose $y$-coordinate is the same as point $B$.

Maria’s Bridge is the collection of all points $P$ obtained by moving the point $A$ along the line $y = 2a$. (Figure 2)

Figure 2. Maria’s Bridge

Using $\theta$, the angle $OA$ makes with the horizontal, as a parameter, we have

$$x = 2a \cot \theta$$
$$y = -a \tan \theta$$

for $0 < \theta < \tan^{-1} 2$, and

$$x = 2a \cot \theta$$
$$y = -a \tan \theta$$

for $\pi - \tan^{-1} 2 < \theta < \pi$. Eliminating the parameter $\theta$, we get the following piecewise mathematical description of the Bridge:

$$y = \begin{cases} 
-\frac{2a^2}{x}, & x < -a \\
2a, & -a \leq x \leq a \\
\frac{2a^2}{x}, & a < x 
\end{cases}$$

Among the interesting properties of Maria’s Bridge:

- It is a continuous curve.
- The function that defines the curve is differentiable everywhere except at $\pm a$.
- Its graph is symmetric with respect to the $y$-axis.
- Its graph is asymptotic to the $x$-axis.
- Its graph has no points of inflection.
- The region bounded by the Bridge and the $x$-axis has infinite area. Given that the area bounded by the Witch and the $x$-axis is finite, this result is perhaps surprising. Some elementary calculus establishes the result.

$$\int_{-\infty}^{\infty} y(x) \, dx = 2 \int_{0}^{\infty} y(x) \, dx = 2 \left( \int_{0}^{a} 2a \, dx + \int_{a}^{\infty} \frac{2a^2}{x} \, dx \right).$$

The last integral in this expression diverges to infinity, since

$$\int_{a}^{\infty} \frac{2a^2}{x} \, dx = \lim_{b \to \infty} \int_{a}^{b} \frac{2a^2}{x} \, dx = \lim_{b \to \infty} 2a^2 \ln \frac{b}{a} = \infty.$$
Agnesi’s Tent

Agnesi’s Tent is constructed by replacing the circle in the Witch of Agnesi with a select isosceles triangle.

1. Let \( P_1 = (-a, 0) \), \( P_2 = (a, 0) \), and \( P_3 = (0, 2a) \) be the three vertices of an isosceles triangle. Draw its sides.
2. Choose any point \( S \) on the line \( y = 2a \) and connect it with a line segment to the origin \( O \).
3. Label the point where the line segment \( OS \) intersects the triangle as \( B \).
4. Let \( P \) be the point whose \( x \)-coordinate is the same as point \( A \) and whose \( y \)-coordinate is the same as point \( B \).

Agnesi’s Tent is the collection of all points \( P \) obtained by moving the point \( A \) along the line \( y = 2a \). (Figure 3)

![Figure 3. Agnesi’s Tent](image)

If we again use \( \theta \), the angle \( OA \) makes with the horizontal as a parameter, we can describe the tent as

\[
\begin{align*}
    x &= 2a \cot \theta \\
    y &= \frac{2a \tan \theta}{2 + \tan \theta}
\end{align*}
\]

for \( 0 < \theta < \frac{\pi}{2} \), and

\[
x = 2a \cot \theta
\]

\[
y = \frac{2a \tan \theta}{-2 + \tan \theta}
\]

for \( \frac{\pi}{2} < \theta < \pi \). When using this parameterization, we will also agree that the point \((0, 2a)\) lies on the curve. Note that

\[
\begin{align*}
    \lim_{\theta \to \frac{\pi}{2}^-} 2a \tan \theta &= \lim_{\theta \to \frac{\pi}{2}^-} \frac{2a \sec \theta}{\sec \theta} = 2a \\
    \lim_{\theta \to \frac{\pi}{2}^+} 2a \tan \theta &= \lim_{\theta \to \frac{\pi}{2}^+} \frac{2a \sec \theta}{\sec \theta} = 2a,
\end{align*}
\]

so from the perspective of continuous functions, the point is precisely where we want it to be.

When we eliminate the parameter \( \theta \), we get the following description of the Tent:

\[
y = \begin{cases} 
    \frac{2a^2}{a-x}, & x < 0 \\
    \frac{2a^2}{a+x}, & 0 \leq x
\end{cases}
\]

Agnesi’s Tent has the following properties:

- It is a continuous curve.
- The function that defines the curve is differentiable everywhere except at \( x = 0 \).
- Its graph is symmetric with respect to the \( y \)-axis.
- Its graph is asymptotic with respect to the \( x \)-axis.
- Its graph has no points of inflection.
- The area of the region bounded by the Tent and the \( x \)-axis is infinite. In fact, the area of the region bounded just by the portion of the Tent that lies in the right half plane and the \( x \)-axis is itself infinite.
\[
\int_{0}^{\infty} \frac{2a^2}{a+x} \, dx = \lim_{b \to \infty} 2a^2 \ln \left( \frac{a + b}{2a} \right) = \infty.
\]

Some remarks

1. Note that this exercise can be adapted for use in a wide variety of classes, ranging from the most elementary courses through calculus classes.

2. Many of our beginning students struggle with the idea of piecewise defined functions. These constructions illustrate that piecewise functions arise quite naturally when solving geometric problems.

3. We have presented only two curves that can be obtained from the Witch of Agnesi. What happens if you replace the circle with a rectangle? Change the triangle from isosceles to equilateral? If you use an inverted parabola? Let your students experiment and see what they discover.

References

Confetti – It’s Not Just for a Party Anymore!

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Many of our students think that math is some kind of a secret, that there is only one way to get the answer and you have to know the “trick” to get that one possible answer. Unfortunately many math teachers encourage these thoughts by providing a limited number of kinds of problems and possible solutions. Students aren’t challenged or encouraged to be creative and think outside the box to use any possible strategy to solve a problem. We have to break that cycle by giving our students problems that challenge their problem solving skills.

A simple overhead projector and a handful of confetti (you can find numerous kinds at the dollar store) provide an endless amount of mathematic possibilities. Have you ever noticed the “magic” that an overhead provides? How a tiny confetti bunny commands attention when enlarged by an overhead? Well, wait until you see the special magic unique shaped confetti infuses into your classroom!

Take the following party hat confetti for example:

Ask students to provide you a possible math problem that is represented by the confetti and remind them they will have to tell what strategy they were using. You might get the answer 5+3 when adding the hats on the top row with hats on the bottom row. You might get 6+2 which represents the dotted hats plus the wavy hats. Hopefully you will also get the commutative property problems for both. Children will want to know which is the “correct” answer so you will have to remind them that since you didn’t establish strict parameters that they are all correct.

Now challenge students to come up with other possible answers. You might get 4+4= hats upside down + hats right side up. You might even get 8+0 = the hats plus the other objects (there aren’t any other objects)! Once students begin to realize that there are several possible solutions they will challenge each other to come up with as many plausible answers as possible.

The next example uses overhead coins. You might get the following answers:

3+4=7 Which could be any of:
coins on the top row + coins on the bottom row
brown coins + silver coins
pennies + nickels
heads coins + tails coins

23 cents
$.03 + $.20=$0.23
Seasonal Confetti

Seasonal confetti or foamy shapes are also a lot of fun. Once your students are pretty good at identifying story problems you can switch and make your own story problems from shapes. In the following example you might ask:

- What is there the most of? Least?
- How many more sleighs are there than bells? (a trick) Bells than gingerbread?
- How many stars do you see? What is an easy way to count the stars?
- How many of these things could you eat?
- How many gingerbread eyes are there?
- If 2 gingerbread could fit in each sleigh, how many total gingerbread could the sleighs hold?
- If each gingerbread was to get 3 bells, how many more bells would you need?

Categorizing/Classifying

Children get a lot of practice identifying attributes through working with pattern blocks and other shapes but it is important to encourage them to think about other ways to sort/order things. Every once in awhile it is fun to put random confetti and see what kinds of things kids will come up with. In the following example they might offer:

- There are 5 different kinds of flowers.
- The star and moon can both be found in the sky.
- The rabbit and butterfly are both animals
- There are two flowers that are congruent.
- There are 15 things all together.
- All shapes but 2 have a line of symmetry.
- 7 shapes have a triangle somewhere within.

Encourage and accept all reasonable explanations.

- How many items all together? Which has the most/least?
- Which items have the same amount?
Other Confetti Uses

Once you get a good collection of confetti you will find many other uses including:

Ordinal Positions

AB, ABC and Repeating Patterns

Subtraction

Expanding Patterns

A Helpful Hint

Purchase a plastic bait box with dividers to keep the confetti separated and organized.

Where Do I Go From Here?

Once your students get creative and have experience looking outside the box they will learn to identify the necessary parameters needed to solve a problem. They will also become adept at using multiple approaches to problem solve. You can now expand that realization into general problem solving. When giving the problem “I need to read seven books this semester. I’ve already read four. How many more do I need to read?” encourage students to share all the possible ways they could get the answer. Would you use a part-part whole box? Draw a picture? Create an algebraic sentence?

Teaching your students to challenge themselves and look at mathematics differently is a difficult but valuable skill. While children are having fun creating stories and “playing” with confetti, they are finding new ways to apply their mathematical knowledge. When your students are no longer limited by traditional problem solving parameters they will begin to apply a variety of strategies to the increasingly difficult problem solving scenarios they will encounter.
Problem Solving in the Primary Classroom

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When I was in second grade, the words math and computation were synonymous. Each day, we completed a worksheet of 30-50 computation problems with one story problem at the bottom of the page. We called it the “dreaded story problem.” Even our teacher called it that! On most days her instructions were to just cross it off because it was too hard or complicated for us to do. On occasion our teacher would be in “a mood” and would require us to complete the problem. So, we would do what any second grader would do. Half of us would add the two numbers together and the other half would subtract the two numbers. Of course, half of us would get the problem right and the other half would get it wrong. When I began teaching math, I realized that the only problem solving we were doing were percentages and fractions because half would get it right and 50% would get it wrong! That was 1968. Unfortunately, not a lot has changed since then. The only difference is that we have an exciting name like, “Problem of the Day” for the story problem.

Most programs do not offer daily multiple opportunities to problem solve, and the problems are usually one step and simplistic. My Action Research Thesis was written about the importance of teaching problem solving in the primary classroom. Through my research I tried to locate and identify some of the highest quality materials available. Unfortunately, after conducting an extensive search of private and textbook companies and scouring material at NCTM conferences, the only resources I was able to find were simple “Problem of the Day” manuals. This became my inspiration to write a book that would meet this need.

Take the following story problem examples:

- Alexandra has 4 cats. Her aunt gives her 3 more cats. How many does she have now?
- Rich has 5 cookies. He eats 2 of the cookies. How many does he have left?

These are typical story problems found in most series today. While we are jumping up and down wondering why our kids aren’t excited about problem solving, these are the questions our students are thinking:

- “Who in their right mind has 4 cats and if you did why would you get 3 more?”
- “What’s wrong with Rich? Any normal kid would have eaten all 5 cookies!”

These kinds of story problems aren’t relevant to children today. They are also too simplistic and can usually be solved by adding or subtracting the two numbers. Instead, we should be teaching our students multiple ways to solve complex and multi-step problems. As adults we realize that there are multitudes of ways to solve real life problems, and if we want our students to be future problem solvers we need them to realize this as well.

I spend 15-20 minutes each day problem solving with my kids. My children quickly learn that it is just as important to be able to tell me how they arrived at the answer as it is to get the correct answer.
Therefore, every student must be able to explain his/her reasoning. After a student models his/her strategy I challenge the rest of the class to come up with a different way to solve the problem. Modeling multiple methods of problem solving provides struggling students with clear examples they might use in the future. It also pushes more advanced students to look farther outside the box.

Take the following example: There are 15 items in my school box. If 4 are pencils, 3 are erasers, and the rest are crayons, how many are crayons?

The first student explains that he used a part, part, part-whole box. He put 15 in the whole and 4 and 3 in the parts. He then knew it was a subtraction problem and once he subtracted 4 and 3 from 15 he would solve for the number of crayons.

The second student explains that she counted up. “I knew I had to get to 15 items so I added the 4 and 3 to get 7. I then used the number line to see how much I needed to add to 7 until I had 15.”

The third student shares that she drew a picture. “I drew the 4 pencils and 3 erasers and then kept drawing crayons until I had 15 items. Then I could count how many crayons I drew.”

Clearly each of these 3 students is working at a different capability of problem solving. All three are going to get the correct answer if given a question like this on a test. Your goal is to assist the student who had to work much too hard drawing all of those pictures by showing her that there are increasingly easier ways to get the correct answer.

We typically solve 4 different types of problems each day. While one Problem of the Day isn’t nearly enough, time limitations and attention spans rarely allow you to work on more than four problems a day.

At the end of this article there is a sample from the book *When Do Dandelions Become Weeds? A Guide to Teaching Problem Solving in the Primary Classroom*. It is a scripted lesson whereby the teacher reads the script and draws (after student responses) what is written in bold – just like the sample on the chalkboard.

I use the scripted lessons with one response for each problem for the first month or so of school so my students can be exposed to a variety of problem solving strategies. After that time, you are ready to accept multiple methods to solve each problem.

Once your students are experienced at identifying multiple ways to problem solve, it is time to move to more complex and multi-step problems. Look at the following samples:

**Different Kinds of Homes**

1. _____people live in an igloo. _____live in a cabin. How many people are living there? Are there more igloo or cabin people? How many more?

2. My apartment building has ____stories. If ____people live on each floor, how many people live in my apartment building?

3. It takes ____stilts to hold up one beach house. How many stilts would it take to hold up ____beach houses?

4. A ski lodge has ____floors. Each floor has one less room than the floor below. If the bottom floor has ____rooms, how many are on the top floor? How many rooms altogether?
The Little Red Hen

1. One bundle of wheat could be ground into ____ cups of flour. How many cups could she get out of ____ bundles of wheat?

2. Little Red can bake ____ loaves each day. How many would that make in a week?

3. Little Red baked _____ white, _____ wheat, and _____ cinnamon loaves. How many loaves is that? Put them in order from most to least? How many more does most have than least?

4. Little Red changes her mind and decides to share her bread with duck, dog and cat. If she cuts a loaf into ____ pieces, how many will each get?

My problem solving topics now center around: current holidays and sports; characters and events in stories we are reading in the classroom; parts of our school day, etc. It is important to keep the topics of high interest to students as the application for our problem solving goals

“Different Kinds of Homes” fits with a basal story and social studies unit where we study different kinds of homes around the world. Problem 1 requires work with number sense. Problem 2 teaches students multiplication through repeated addition. Problem 4 is a much more complex problem where students might have to draw a diagram or create a formula to solve.

“The Little Red Hen” takes a childhood classic and requires students to work with number sense, multiple patterning, and fractions. Questions 3 and 4 even provide discussion opportunities on what kind of bread do students think she really did bake and who thinks that when she shared it was the right thing to do?

Notice that the number amounts are purposefully left blank for two reasons. You know your students best and can choose numbers that would be appropriate for the level of problem solving they are ready to complete. The other reason is that it gives you an opportunity to teach your students how basic problem solving skills will help them as their problems become more difficult and the numbers get larger. Take Little Red Question 2 as an example. If I taught this lesson early in the week, I might put a 2 in the blank. As we problem solved, my students would discover that using our doubles or counting by twos would be an easy way to get the answer. If I re-taught this lesson at the end of the week I might put the number 8 in the blank. We could initially use our doubles to get started and then use a hundred chart to add the doubles together. Another fun thing to do is let students choose the numbers the second time you problem solve. When someone chooses 99 (thinking it will be funny) you can actually challenge students to find an easy way that the problem could still be solved.

Problem solving is a lot of work for the teacher and for your students. You will begin this journey with the goal of raising student achievement in problem solving. While you will easily accomplish this goal, you will also discover that you and your students are now experienced, successful AND eager problem solvers.

About the Author
Rena Pate is a National Board Certified Teacher. She has a BS degree from the University of Illinois and a Master’s in Teaching and Learning Mathematics. She has taught first grade for 23 years in Danville, Illinois.

She has presented at several ICTM workshops, regional NCTM in New Orleans and St. Louis and the Annual NCTM in
Indianapolis. You can see her presentation at the regional NCTM in Chicago, Illinois, November 29th. For more information, visit her website at primarymathrules.com.

Below is a sample from the book *When Do Dandelions Become Weeds? A Guide to Teaching Problem Solving in the Primary Classroom* by Rena Pate.

### Day 26

**Draw a part-part whole box.** We had 15 kick balls. The bigger kids kicked 4 of them onto the roof. How many balls do we have left? If 3 classes had to share the remaining balls, how many would each get? How many would be leftover?

---

**My candy jar has 3 gumdrops, 5 lemon drops, and 8 gummy worms. Record initials and numbers.** How many pieces of candy do I have? Put them in order from greatest to least. If I cut my gummy worms in half how many pieces would I now have? If I am allowed to eat 2 pieces a day, how many days will my candy last me?

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**Record the letters APT and the values 2, 3, 4 on the board.** Who can make a word out of these letters? Let’s take that word and repeat it 2 times. **Transfer the values under the letters.** What can we do to make it easier to add these together? Can someone make a different word? See if they know PAT and TAP patterns would be equal? What about a 2 letter word?

---

**Dan, Sam, Kam and Pam are collecting lightning bugs. Record their names on the board.** Sam collected 6. The other children have collected 10, 4 and 7. Figure out who has which amount. Dan has the most. Pam has less than Sam. Kam has less than Dan.
401 102 uses for a paper clip

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Paperclips have many uses besides just clipping papers together. This simple piece of twisted metal can be used in over 100 ways, for example: a picture hanger, an emergency fish hook, a book mark, an ejector for a powered-off CD-ROM, or a zipper pull replacement. Even though there are plenty of unique ways to use a paperclip, I have managed to find one more—compass and straightedge constructions.

From my experience, students learning compass and straight edge constructions can become overwhelmed when trying to remember the various steps they must follow. For many students the most frustrating part of the process is wrestling with a compass that slips, pokes holes in their paper, and is a general nuisance. I have developed an activity that allows students to perform most compass and straightedge constructions without a compass.

I typically use this activity as a review of constructions, prior to an assessment. I find that in this way, even students who have mastered using a compass benefit from the activity. In this article, the constructions that are emphasized are the bisection of a short line segment, bisection of an angle, construction of a perpendicular line through a point on a given line, and the bisection of a very long line segment.

In this activity, the compass is replaced by a paper clip and two writing utensils. A jumbo size paper clip is recommended, although a standard size paper clip will suffice. The paper clip lies flat on the paper with a pencil inside the paper clip at both ends, perpendicular to the paper. One pencil is held still, as a pivot, and the other is used to draw the arcs, as shown in Figure 1.

The markings on the paper will look just like the arcs and lines made in traditional constructions. I find that if I demonstrate just one construction with a paper clip, students grasp the concept and are able to complete the activity with minimal assistance.

The simplest constructions to do are the following:

- the perpendicular bisector of a (short) segment
- an angle bisector of a angle
- an equilateral triangle
- a line perpendicular to a given line through a point on the line
- a line perpendicular to a given line through a point not on the line (but close to the line)
- a line parallel to a given line through a point not on the line (but close to the line)
- the circumcenter and incenter of a (small) triangle
- the altitudes and medians of a (small) triangle

I will describe a few of the constructions here, but anticipate that the rest will be self-evident. I will begin by constructing a...
perpendicular bisector of a short line segment. For the purpose of the example, consider line segment $AB$, which is 6 cm in length (Fig. 2).

![Fig. 2](image)

Place one end of the paper clip at point A and make an arc (Fig. 3).

![Fig. 3](image)

Now, move the paper clip to point B and make an arc that intersects the first arc. Use a straight-edge to connect the intersection points of the arcs (Fig. 4).

![Fig. 4](image)

This should seem like a fairly straightforward compass and straight-edge construction.

The next construction is slightly more complex. Consider the bisection of an angle, such as angle $A$ (Fig. 5) which, for illustrative purposes, I have made an angle of 160°. The basic idea of the construction remains the same as in the traditional construction: place the paper clip at the vertex of the angle and make an arc that intersects the rays of the angle.

![Fig. 5](image)

Place the paper clip at each intersection point, make arcs, and connect the vertex with the intersection point of the arcs. We recommend that the first arc be produced using the inner loop (Figs. 6 and 7), or it may be difficult to find the intersection point of the last two arcs (Fig. 8).

![Fig. 6](image)

![Fig. 7](image)

In Figure 9, I show the result of the construction using the inner loop. While it is possible, albeit messy, to bisect an angle

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using only the outer loop, for the next construction the inner loop must be used.

Fig. 9

The third construction is that of a line perpendicular to a given line, through a point on the line (Fig. 10).

Fig. 10

The first step in this construction is to make an arc centered at point P that intersects the sides at points A and B. Next, make arcs centered at points A and B, which should intersect each other above and below the line. Note that if only the outer loop is used for all three arcs, the last two arcs will intersect each other at point P, rather than above and below the line (Fig. 11).

Fig. 11

The inner loop is used for the first arc (Fig. 12), producing points A and B, and the outer loop is used for the last two arcs. This will result in an image similar to Figure 13.

Fig. 12

Fig. 13

The previous constructions were simple; now we will look at a final, more complex, construction. A construction, which at first glance, appears to be impossible to do with a paper clip, turns out to be merely challenging. Consider bisecting a line segment which is 12.5 cm long. In the traditional compass and straight-edge construction, one opens the compass to slightly more than half the length of the line segment. However, using a jumbo size paper clip, the longest arc that can be made is slightly less than 4.5 cm. Thus, one might jump to the conclusion that it is not possible to bisect this line segment using the paper clip method. However, it can be done.

Let $AB$ be a line segment 12.5 cm in length. Use the paper clip to make arcs centered at points A and B; let C and D be the points where these arcs intersect $AB$ (Fig. 14).
Now, construct the perpendicular bisector of $\overline{CD}$, which is also the perpendicular bisector of $\overline{AB}$. Note: this idea can be used to construct the perpendicular bisector for segments of any length.

**Conclusion**

Students are often mystified by compass and straightedge constructions. The simplicity of a paper clip, in comparison to a compass, seems to make some students more comfortable with geometric constructions. In fact, I will occasionally observe a student opt for a paper clip over a compass, when given the choice.

Another alternative to a compass would be to use a string. You can tie a knot, creating a loop, similar to the paperclip. We invite you to find other creative ways to do constructions.

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**Guidelines for Submitting Manuscripts**

Manuscripts should be no more than 3000 words in length and should be prepared in either Microsoft Word or LaTeX. Be sure to provide accurate and complete bibliographical information with references listed at the end of the manuscript. Articles to be published will be formatted by the editors for uniformity and converted to PDF. Authors will receive page proofs for their approval before final publication.

No information that could help a reviewer identify the author (including your name or school) should appear in the manuscript. This is to maintain the integrity of the blind review process. This information will be inserted should your manuscript be accepted for publication.

Proofread your manuscript carefully before submitting it, reviewing for grammar, mathematical correctness, and completeness and accuracy of references. Do not submit manuscripts that have previously been published or are under consideration for publication elsewhere.

Submissions should be sent electronically to the email address below. Your email should include your name, position, work affiliation, phone number, and city and state. Attach to your email the Word or LaTeX file (both source file and pdf) containing the text of your manuscript. For any photographs or figures that appear in the manuscript, attach each figure as a separate file using the highest resolution possible, even if those images are also embedded with the manuscript. You may also, if you wish, attach a photo of yourself for publication alongside your article.

Submit publication request to [imt@ictm.org].
Students begin learning about transformations in the early elementary grades. They might create simple tessellations by cutting and pasting cardboard pieces, or investigate symmetry by folding figures on patty paper (Common Core State Standards [CCSS] 4.G.3). Their knowledge and ease with transformations continues to grow through middle school, where the everyday language of turn, flip, slide, bigger and smaller is connected with the corresponding mathematical terminology of rotation, reflection, translation, and dilation (National Council of Teachers of Mathematics [NCTM] 2000). By the end of middle school, students should be able to describe the effect of transformations on two-dimensional figures (CCSS 8.G.3).

In high school, however, students often study transformations in isolation, establishing few connections with other parts of geometry. This disconnects the informal thinking of elementary and middle school, and the deductive reasoning required in a high school geometry course. Coxford and Usiskin (1971, 1975) originally proposed the use of transformations to build conceptual understanding based on prior experiences before deriving proofs. Transformations also form the basis of geometric understanding in the vision of the Common Core. For instance, for high school geometry “[t]he concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation” (National Governors Association for Best Practices & Council of Chief State School Officers, 2010, Geometry: Introduction section, para. 4).

Using transformations to strengthen your students’ understanding of geometry sounds like a good idea, but how can you actually do this in the classroom? This article describes geometry activities that incorporate transformations to discover and justify definitions and properties of quadrilaterals, and the relationships between the different quadrilaterals. We provide examples how you can help students use transformations in their reasoning. The lesson described in this article incorporates CCSS geometry content from across the grades. Specifically, it addresses the following standards in the Congruence domain (G-CO):

<table>
<thead>
<tr>
<th>Experiment with transformations in the plane</th>
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<tbody>
<tr>
<td>G-CO.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.</td>
</tr>
<tr>
<td>G-CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.</td>
</tr>
</tbody>
</table>
Completing the activities presented in this article will help your students develop the following mathematical practices [CCSS MP]:

1. Make sense of problems and persevere in solving them;
3. Construct viable arguments and critique reasoning of others;
5. Use appropriate tools strategically.

Blackline masters for all activity pages are included at the end of this article. Examples of student work and suggestions on how to deliver the material can be found throughout the text.

Discovering and developing an understanding of quadrilateral properties

As mathematics teachers, we know that simply providing students with a list of definitions and properties of quadrilaterals to memorize is an ineffective practice. This approach leads to minimal understanding of the quadrilateral properties and how they are derived. In addition, little or no connections are made between the properties and how the quadrilaterals relate to each other. Transformations give students the opportunity to develop the definitions and properties of quadrilaterals through discovery and to build connections among the family of quadrilaterals. In the activity we describe, students create quadrilaterals using transformations of triangles and discuss how to derive the properties of the resulting quadrilateral. With teacher guidance, students synthesize and summarize their findings and investigate how these properties relate the different quadrilaterals to each other. Students then develop definitions to classify each quadrilateral shape accordingly (CCSS MP 3). The outline for this activity is as follows:

1. Form heterogeneous groups of 3 - 4 students.
2. Assign each group one or two quadrilaterals to investigate parallelogram, rhombus, rectangle, square, trapezoid, isosceles trapezoid, or kite.
   a. Give each student in the group an investigation sheet to record their findings. Blackline masters for these six investigations can be found at the end of the article.
   b. In addition, give each group a sheet of graph paper, a ruler, and a sheet of patty paper, which students can use to complete the prescribed rotations in some of the investigations (CCSS MP 5).
3. During the investigation phase of the lesson, each group creates a quadrilateral by completing a set of transformations on a given triangle. In the process, the students discuss the various properties of the quadrilateral shape they create with respect to the sides, the vertex angles, the diagonals, and the symmetry of the quadrilateral.
4. Students are asked to go back and discuss how they could justify the various properties using their prior knowledge of transformations (CCSS MP 3). In particular, students should have developed the concept that rotations, reflections, and translations preserve congruency of segment lengths.
as well as angle measures in an earlier lesson.

5. The students determine how the quadrilaterals relate to each other by creating a family tree for the quadrilaterals (see the last blackline master). In the process of creating a family tree, students formulate a definition of each shape, making sure each definition specifies minimum properties while at the same time excluding any unwanted cases.

An alternative to number 5 above is the use of expert and jigsaw groups. Using this method, students are first grouped into expert groups where they create a single quadrilateral and record its properties. Then new jigsaw groups are formed which contain at least one expert for each quadrilateral. Every member shares with the jigsaw group the properties of their particular shape. Then each jigsaw group creates a quadrilateral family tree.

Justifying definitions and properties using transformations

Discussion provides an opportunity for the teacher to guide students in the use of deductive reasoning to prove quadrilateral properties. We describe two instances of student reasoning with transformations as an example of how to facilitate this type of discussion. The first group constructed a parallelogram (see Figure 1).

When asked what they noticed about the vertex angles of the parallelogram they created, the students responded that the opposite angles were congruent. The teacher prompted the students to provide a justification for their reasoning (CCSS MP 3). In response, the students simply stated that they measured the angles of the parallelogram and found that one pair of opposite angles measures 40 degrees while the other pair of opposite angles measured 140 degrees (see Figure 1). The teacher asked the students if they would be able to draw the same conclusion if the original triangle they started with had different angle measures.

At this point, students were struggling to move beyond the specific parallelogram they had created and think in terms of any arbitrary parallelogram. The teacher reminded the students that simply measuring the angles did not constitute a justification that the property would hold true for all parallelograms. The students were instructed to think back to the properties of transformations they had discussed in an earlier lesson. One of the students in the group now stated (CCSS MP 1) that when they started with the original triangle, the vertex angle (marked 40 degrees on their paper) would always be congruent to the opposite vertex angle because they rotated the original triangle to form the parallelogram. Since rotations always preserve congruency, the opposite vertex angles would be congruent. Using single and double notation marks for the two angles, which formed the other vertex angle of the parallelogram, the students were able to justify that the other pair of opposite vertex angles also had to be congruent by the fact that rotations preserved congruency. The group summarized their knowledge of the properties of parallelograms (see Figure 2).

Figure 1. Student drawing from group discussion of parallelograms

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The second group in another class started their construction of a rectangle with right \( \triangle ACB \). They found the midpoint \( M \) on \( AB \), drew the median \( MC \), and then rotated the triangle 180 degrees around midpoint \( M \) (see Figure 3).

Figure 3. Student drawing of rectangle \( ACBC' \)

Measuring the sides and the angles, the students marked congruent sides and angles in the figure. One of the questions that arose in this group was whether \( \overline{MB} \) would always be congruent to \( \overline{MC} \). The students felt that this would always be true, but were unsure why. When prompted by the teacher to think back to the relationship between the sides and angles of triangles, the students realized that if they could show that the base angles of \( \triangle CMB \) were congruent, and then the sides \( \overline{MB} \) and \( \overline{MC} \) opposite those angles would be congruent.

The question thus became how to show that \( \angle C'CB \cong \angle CBA \). First, students noticed that \( \angle CAB \) is complementary to \( \angle CBA \), because these are the base angles of original right \( \triangle ACB \). Realizing that \( \angle CBA \) corresponds to \( \angle C'B'A \) under the rotation, they concluded that the angles are congruent. The students concluded that \( \angle C'BC \) is a right angle. Then they argued that \( \overline{AC} \cong \overline{BC'} \) and \( \overline{BC} \cong \overline{AC'} \) because the length of a segment is preserved under rotation. This allowed the students to say that \( \triangle ACB \cong \triangle BC'A \). Hence, they knew that \( \angle C'CB \cong \angle CBA \) because the two angles are corresponding parts of congruent triangles, which proved that \( \overline{MB} \cong \overline{MC} \). To deepen students’ understanding, the teacher prodded the students to make explicit connections to the side-angle-side triangle congruence postulate.

Using their prior knowledge of transformations and applying this knowledge to a series of investigations, students gained a deeper understanding of the properties of the quadrilaterals and were able to justify the properties. Although the students used transformation in their reasoning, their final justification is no less rigorous than a traditional two-column proof. This activity was highly engaging and had the benefit of students asking questions about the properties conjectured.

In one particular classroom, students demonstrated a deeper understanding of the quadrilateral properties by creating a quadrilateral family tree. Although more
time was spent discovering and deriving properties than is customary, students successfully demonstrated their understanding by constructing the family tree for the quadrilateral shapes. The variations in student representations were evident in the family tree constructions.

Some groups organized their quadrilateral shapes in a bottom-up fashion starting with the most specific shape, the square, and working back up to the more general quadrilateral. Other groups organized their family tree using a top-down method with the more general parallelogram figure at the top and working down to the more specific figures (see Figure 4).

Figure 4. A top-down quadrilateral family tree

![Quadrilateral Family Tree Diagram](Tree Photo from FreeDigitalPhotos.net)

Most groups had an easier time placing the kite and the trapezoid in the family tree diagram than the rest of the figures. At the center of the discussions was the relationship between the rectangle, rhombus, and square. After the teacher directed students’ attention to the properties that were similar across the figures, the students correctly concluded that every square has the properties of both a rhombus and a rectangle.

In summary, the lesson can be delivered as follows. At the beginning of the lesson, students investigate the properties of a particular quadrilateral in groups. In a larger class, you may need to assign the same quadrilateral to different groups; in a smaller class, you could assign each group more than one quadrilateral, or you could assign only a subset. In the first stage of the investigation, students most likely focus on discovering the properties. In the second stage, direct the attention of the students to explaining and justifying these properties. After students have completed their investigation, you may want to bring the class together to discuss the similarities and differences in the properties discovered. This also provides students an opportunity to use mathematical arguments to justify their conclusions, and to judge the validity of arguments made by their peers. Finally, students make connections between the different quadrilaterals by constructing a family tree. When using this lesson in your classroom, you may want to keep the following suggestions in mind:

- If a group of students initially fails to come up with any properties, you may need to use a question to point them in the right direction. For instance, “What do you notice when you measure the vertex angles?”
- Once a group of students has discovered a property, guide the students in constructing a justification using transformations.
- If a group of students finds it difficult to generalize from their drawing of a specific parallelogram (or another quadrilateral) to the general case, you may want to suggest drawing another parallelogram.
Advancing geometric thought through the use of transformations

The van Hiele levels of geometric thought (1986) provide a sound justification of why transformations are a good bridge between initial student thinking and the abstract nature of high school geometry. The model includes five levels of geometric reasoning: visual, descriptive, informal deductive, (formal) deductive, and rigor. Although a student progresses through the levels linearly, there is no strict dependence on age. For example, a fourth grader and a high school geometry student could be at the same level. The levels overlap as a student transitions from one level to the next. Most students entering a high school geometry course operate at or below the van Hiele visual level of understanding (Shaughnessy and Burger 1985). These students are able to identify different shapes, but they may find it hard to recognize and reason with specific characteristics of shapes. For example, doing transformations with everyday language as described in the introduction is at the visual level. Students at this level are often not yet ready for the abstract reasoning required in high school geometry. Students using the mathematical terminology for transformations are at the overlap of the visual and descriptive levels. Students discovering and deriving quadrilateral properties are at the informal deductive level.

Transformations provide a bridge between the initial visual intuition of the students and the more formal reasoning of the higher van Hiele levels. The quadrilateral activities provided entry to students at the descriptive level. Students drew conclusions about the sides and angles of their original triangle and then rotated the triangles to create the specified quadrilateral. This provided students the opportunity to apply prior experiences of transformations and the properties of triangles. With the discussions, the teacher guided students to the informal deductive level by having them discover the properties of quadrilaterals and create informal arguments deriving the quadrilateral properties. Finally, students unify their understanding of quadrilaterals by creating the family tree.

Conclusion

Incorporating transformations to develop understanding in high school geometry helps students move from the visual and descriptive van Hiele levels to the informal deductive and formal deductive levels. All students, regardless of their van Hiele level, stand to benefit from an integration of transformation geometry in a high school geometry course (Usiskin 1972; Okolica and Macrina 1992). In the quadrilateral activities described above, the teacher used this approach to guide the students in building their understanding of quadrilateral properties. When students can connect geometric concepts to the prior knowledge and experiences that they bring to class, they can formulate deductive arguments. The study of transformations in the early grades and the continuing development of these concepts in middle and high school can provide the bridge geometry students need to reach the informal and formal deductive levels of geometric reasoning.

References


Blackline masters for the activity pages are included below and on subsequent pages.

**Building Quadrilaterals: Parallelogram**

1. On a sheet of graph paper, draw obtuse $\Delta ABC$.
   - Draw one of the sides of $\Delta ABC$ along one of the grid lines.
   - Be sure all vertices are placed at the intersection of grid lines.

2. Locate the midpoint, $M$, of $AB$.

3. Draw the median to side $AB$.

4. Label the figure you have drawn by indicating congruent sides, angles, and measures using appropriate markings.

5. Rotate $\Delta ABC$ by $180^\circ$ around point $M$.
   - Label the new image with the correct markings to indicate congruent sides, angles, and measures.

6. Discuss with your group the properties of the parallelogram that you created using the transformations above. Pay attention to the properties of the sides, the vertex angles, the diagonals, and the symmetry of the figure.
   - Summarize your findings under the headings on the chart.
Building Quadrilaterals: Rectangle

1. On a sheet of graph paper, draw scalene right \( \triangle ABC \).
   - Draw both legs of \( \triangle ABC \) along grid lines.
   - Draw the right angle at vertex \( C \).
   - Be sure all vertices are placed at the intersection of grid lines.

2. Locate the midpoint, \( M \), of \( AB \).

3. Draw the median to hypotenuse \( AB \).

4. Label the figure you have drawn by indicating congruent sides, angles, and measures using appropriate markings.

5. Rotate \( \triangle ABC \) by 180° around point \( M \).
   - Label the new image with the correct markings to indicate congruent sides, angles, and measures.

6. Discuss with your group the properties of the parallelogram that you created using the transformations above. Pay attention to the properties of the sides, the vertex angles, the diagonals, and the symmetry of the figure.
   - Summarize your findings under the headings on the chart.

Building Quadrilaterals: Rhombus

1. On a sheet of graph paper, draw scalene right \( \triangle ABC \).
   - Draw both legs of \( \triangle ABC \) along grid lines.
   - Draw the right angle at vertex \( C \).
   - Be sure all vertices are placed at the intersection of grid lines.

2. Label the figure you have drawn by indicating congruent sides, angles, and measures using appropriate markings.

3. Reflect \( \triangle ABC \) across the line containing \( BC \).
   - Label the new image with the correct markings to indicate congruent sides, angles, and measures. Use prime marks for the image vertices.
   - What kind of figure do you have now? Justify your answer.

4. Reflect \( \triangle ABA' \) across the line containing \( AA' \). Be sure to also reflect \( BC \).
   - Label the new image with the correct markings to indicate congruent sides, angles, and measures. Use prime marks for the image vertices.

5. Discuss with your group the properties of the parallelogram \( ABA'B' \) that you created using the transformations above. Pay attention to the properties of the sides, the vertex angles, the diagonals, and the symmetry of the figure.
   - Summarize your findings under the headings on the chart.
Building Quadrilaterals: Square

1. On a sheet of graph paper, draw isosceles right $\Delta ABC$.
   - Draw both legs of $\Delta ABC$ along grid lines.
   - Draw the right angle at vertex $B$.
   - Be sure all vertices are placed at the intersection of grid lines.

2. Label the figure you have drawn by indicating congruent sides, angles, and measures using appropriate markings.

3. Reflect $\Delta ABC$ across the line containing $\overline{BC}$.
   - Label the new image with the correct markings to indicate congruent sides, angles, and measures. Use prime marks for the image vertices.
   - What kind of figure do you have now? Justify your answer.

4. Reflect $\Delta ACA'$ across the line containing $\overline{AA'}$. Be sure to also reflect $\overline{BC}$.
   - Label the new image with the correct markings to indicate congruent sides, angles, and measures. Use prime marks for the image vertices.

5. Discuss with your group the properties of the square $A'C'A'C$ that you created using the transformations above. Pay attention to the properties of the sides, the vertex angles, the diagonals, and the symmetry of the figure.
   - Summarize your findings under the headings on the chart.

Building Quadrilaterals: Kite

1. On a sheet of graph paper, draw scalene acute $\Delta ABC$.
   - Draw side $\overline{BC}$ of $\Delta ABC$ along a grid line.
   - Be sure all vertices are placed at the intersection of grid lines.

2. Draw an altitude, $\overline{AD}$, of $\Delta ABC$ from point $A$ to $\overline{BC}$.

3. Label the figure you have drawn by indicating congruent sides, angles, and measures.

4. Reflect $\Delta ABC$ across the line containing $\overline{BC}$.
   - Label the new image with the correct markings to indicate congruent sides, angles, and measures. Use prime marks for the image vertices.

5. On a separate sheet of graph paper, repeat steps 1-4 with a scalene obtuse $\Delta ABC$.

6. Discuss with your group the properties of the kites $A'CA'B$ that you created using the transformations above.
   - Summarize your findings under the headings on the chart.
Building Quadrilaterals: Trapezoid 1

1. On a sheet of graph paper, draw a large scalene $\triangle ABC$.

2. Label the figure you have drawn by indicating congruent sides, angles, and measures using appropriate markings.

3. Locate a point $T$ anywhere on $AB$.
   - Construct a line parallel to $BC$ through point $T$.
   - Construct the intersection of this line with $AC$ at point $R$.

4. Discuss with your group the properties of the trapezoid $TBCR$ that you created using the transformations above.
   - Summarize your findings under the headings on the chart.
   - Hint: What connections can be made between the measures of the angles in this figure and your prior experiences with parallel lines cut by a transversal?

Building Quadrilaterals: Trapezoid 2

1. Now draw a large isosceles $\triangle XYZ$.
   - Draw the vertex angle at $X$.

2. Locate a point $A$ anywhere on $YX$.
   - Construct a line parallel to $YZ$ through point $A$.
   - Construct the intersection of this line with $XZ$ at point $B$.

3. Discuss within your group the properties of the trapezoid $AYZB$ that you created using the transformations above.
   - Summarize your findings under the headings on the chart.
   - Hint: What connections can be made between the measures of the angles in this figure and your prior experiences with parallel lines cut by a transversal?
Names

Properties of

<table>
<thead>
<tr>
<th>SIDES:</th>
<th>VERTEX ANGLES:</th>
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<th>DIAGONALS:</th>
<th>SYMMETRY:</th>
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Quadrilateral Family Tree

Your group will be building a family tree for all of the quadrilateral shapes that you have explored so far in this lesson.

Your family tree will be a diagram that connects all the different quadrilateral shapes together based on their shared properties. When designing your family tree it would be best to start with the more generic figures at the top of the diagram. These would be the figures whose properties were a part of all the other shapes. You will use the shapes that were cut out for the “Where Do I Belong Activity.”

As you come to more shapes with more specific and unique properties, you will need to place these underneath the first set of figures. Be sure to show connections between figures using line segments that branch between different quadrilaterals.

Your family trees should branch off from the most generic name for a 4-sided figure: quadrilateral. Once you have decided for sure what your family tree should look like, glue or tape the shapes in their appropriate place.

Quadrilaterals Family Scrapbook

Using paper, pencil, and pictures or photos, your group needs to design a scrapbook for the family of quadrilateral shapes. Your entire group will be responsible for creating one scrapbook. Each person in the group must create at least one page of the scrapbook.

Each shape must have at least one page in the scrapbook. Each page should contain at least the following:

- A picture of the quadrilateral shape seen in the real world
- A list of the characteristic properties of that particular shape

In addition, the scrapbook needs to have a copy of the quadrilateral family tree that was created in class.

If your group chooses, you may also complete the scrapbook in PowerPoint on the computer.
**ICTM MEMBERSHIP APPLICATION FORM**

Clip out this page and mail it with your payment to the address below.

![ ] New Member  ![ ] Reinstatement  ![ ] Renewal  ![ ] Change of Address

Name _______________________________  Member Number __________________

Check preferred mailing address. Please complete BOTH columns.

**Home**

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Regional Office of Education

NCTM Member?  ![ ] Yes  ![ ] No

**Profession: (check only one)**

- [ ] EC-3 Teacher
- [ ] 4-6 Teacher
- [ ] Jr. High/Middle Teacher
- [ ] Sr. High Teacher
- [ ] Special Education Teacher
- [ ] Community College
- [ ] College/University
- [ ] Administration
- [ ] Retired
- [ ] Student
- [ ] Institutional Member
- [ ] Other

**Interests: (check up to three)**

- [ ] Remedial
- [ ] Gifted
- [ ] Teacher Education
- [ ] Assessment
- [ ] Certification
- [ ] Multicultural Education
- [ ] Teacher Evaluation
- [ ] Professional Development
- [ ] Scholarship
- [ ] Technology
- [ ] Research
- [ ] Math Contest

**Dues for ICTM Membership:**

- **Regular member**
  - ![ ] one year $35
  - ![ ] *three years $90
  - *(This special rate expires 3/31/2010)*
  - ![ ] five years $160

- **Retired Member**
  - ![ ] one year $30

- **Student Member**
  - ![ ] one year $20

- **Institutional Member**
  - ![ ] one year $100

SPECIAL OFFER:
Between April 15, 2009 and March 31, 2010, purchase a three-year membership for only $90. After March 31, 2010, the price for a three-year membership will be $100.

Please note, the ICTM membership year ends on November 5, and memberships are not prorated. However, memberships purchased between April 1 and November 4 will be active for the full membership cycle purchased, PLUS a grace period between the date of purchase and November 5, the beginning of the next full membership cycle.

If recruited as a new member by a current member, please list the recruiter’s name ________________________________

Mail this application and a check or money order payable to: **EASTERN ILLINOIS UNIVERSITY**

ICTM Membership
School of Continuing Education
Eastern Illinois University
600 Lincoln Avenue
Charleston, IL  61920-3099

Total Enclosed: $ __________________