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From the Editors…

Welcome to the Fall 2009 issue of the *Illinois Mathematics Teacher*. It has been awhile, but we now have a healthy selection of articles to share with the members of ICTM.

This issue of the *IMT* contains general interest articles as well as articles specific to a variety of levels of student learning. Carol Castellon and Warren Buck have written an article about why some students struggle when trying to understand their teachers. Jeong Oak Yun and Alfinio Flores have an article about using polygonal numbers to foster algebraic thinking. There is a very interesting project for students at the precalculus and calculus level concerning tumor reduction in cancer patients. If you are introducing fractions at the elementary level, “S-Q-W-O-R-M-Y Fractions” is a great activity for the classroom. “Isosceles Pentagon” includes some interesting problems for your geometry students. Paula R. Stickles provides a great article on using concept maps with preservice teachers. Cheng-Yao Lin looks at several techniques for mental calculations. William M. Carrol looks at using practical applications in a trigonometry classroom. “Fibonacci Pattern Formulas” includes several patterns of interest as well as activities for students to try in your classroom.

Typically included at the end of each issue is a form for becoming a reviewer for the *IMT*. This form has not been included in this issue due to page considerations. If you are interested in becoming a reviewer, please email tvoepel@siue.edu and list the subject or grade levels you would be interested in reviewing. In order to ensure that the articles are of interest to our readers, we send them to reviewers to get their approval. We would love to have reviewers willing to review in only one area or multiple areas. As postage prices continue to rise, we are trying to conduct most of our communications by email, so please update your information and your email address if you have not reviewed in the past year.

We would appreciate hearing from any of you out there who are reading the journal. We would especially like to hear about any activities from the *IMT* that you have used with your students. Your comments and constructive criticism are heartily solicited.

Please consider submitting an article or classroom activity to the *IMT*. Consider writing about an activity that you use in your classroom and you would like to share with others. Your articles are needed to continue the sharing of ideas and the publishing of this journal on a regular basis.

Thank you for sharing.

*Marilyn and Tammy* editors
“A Commentary on the Importance of Using Correct Mathematical Terminology” or “One Reason Why Students May Have Trouble Understanding College Professors”

Carol Castellon and Warren Buck – University of Illinois, castelln@illinois.edu

“I think we need a few more of those thingies to support the bridge,” said the structural engineer. “Why don’t we put some of those curved do-jiggies above the windows?” posed the architect. “Some of your number-deals are high . . . here, I’ll give you an Rx for some stuff,” prescribed the doctor. Have you ever considered how difficult it would be to communicate, even at a basic level, without using the right vocabulary?

In recent years, teachers have explored several non-traditional ways of teaching mathematics. Increased emphasis on group learning and students exploring topics on their own make it possible for students to get through a K-12 mathematics curriculum without using mathematical terminology or hearing a teacher speak using correct mathematical vocabulary. One downside of group learning on a daily basis is that students do not need to use math terminology to communicate with their learning group to solve problems or complete the activities.

Mathematics is, among other things, a language. A student has to understand the vocabulary of math to understand math. In other words, a student has to recognize and understand the terms being used to grasp the topics being taught. Both the textbook and the teacher must use mathematical terminology consistently. Repeated often, the terms will be understood well enough to insure that communication between students, teachers, and textbooks leads to success in understanding mathematics.

Students can hear a word being used, and they may have read the definition in a book; however, unless they actually hear it being used in context by a teacher, it is unlikely that they will understand the term or use it when they do the math. Our experience with students in public schools, community colleges, and now at the University of Illinois, reveals that many students who say they do not understand the math, often do not comprehend the math terminology well enough to understand our explanations. Understanding terminology is the foundation of success in math and many other disciplines. This understanding comes from seeing and hearing terms being used in context rather than reading a definition in a book. The words have to be part of a student’s working vocabulary, or a communication gap is created and learning stops.

A leading text for teacher education argues the advantages of using the terms top number and bottom number rather than numerator and denominator. Noticeably missing in similar math texts for teacher education are vocabulary terms such as addends, minuend, product, quotient, coefficient, roots, vertex, chord, perimeter, reciprocal, digits, and integer. Not only can use of improper mathematical terminology lead to problems in communicating mathematical ideas, it can also be somewhat offensive. For instance, a colleague spoke of “Dolly Parton fractions” as fractions that are heavy on the top, rather than using the term improper fractions. While it is true that teachers can teach a math curriculum
and avoid math terminology, it is unlikely this approach to teaching math will increase the number of math majors at our universities.

We have found that students who have a poor mathematical vocabulary dislike math because they think that they cannot do it well. In reality, they could do math well if only they understood the mathematical terms we use in our explanations. Try to imagine teaching complex analysis or differential equations using a 3rd grade vocabulary. Moreover, even if they recognize the terminology, they may not totally grasp the concept that the term defines because they have not heard and seen the term used in context. For instance, most students hear the words area and ratio, but some of our students have no intuitive understanding or working knowledge of either. Unfortunately it seems that, by the time some of these students reach college, they are so disinterested in math that they are unwilling to learn the vocabulary and simple concepts that would enable them to understand math. This is a vicious cycle that elementary school teachers, high school teachers, and even authors of mathematical texts can address if they would make more of an effort to always use correct mathematical terminology. An explanation of how these terms relate to real life situations would help students understand the meaning of those terms.

We would agree that, in most cases, a student’s lack of understanding mathematics is attributed to a poor conceptual foundation, but at least part of the problem may be a communication gap from an incomplete math vocabulary. For instance, while working with an undergraduate, we used the term squared to explain a problem. The student asked what we meant by squared and we replied “as in 6 squared,” . . . still a blank look . . . “like 6 times 6.” “Oh yeah,” she said. Her understanding of the term times was better than her understanding of squared. Equally alarming are educators who believe and teach that generic words have only one meaning in math. For example, altogether “always means add,” and they would argue if the average height of boys is 44 inches and the average height of girls is 46 inches, then the average height of the children altogether must be 90 inches. Have you ever heard someone say guzinta (as in 3 guzinta 6 two times) or times it (instead of multiply)? Recently, a student asked what a quotient was, and thankfully understood when we responded, “answer to a division problem.” We could not bring ourselves to use the word guzinta to explain quotient. While math educators joke about this jargon, college professors usually assume their students have acquired a mathematical vocabulary, and probably do not or cannot take class time to define fundamental terminology. College instructors will refer to the numerator rather than the top number, and never consider that students might not recognize the word numerator, as the student has always heard the term top number.

The three unanswered questions we are posing are: (1) To what extent is a student’s lack of understanding college mathematics attributable to a poor working vocabulary, and to what extent to a poor conceptual foundation? (2) To what degree are vocabulary and conceptual understanding intermingled; i.e., how important is a mathematics vocabulary to success in college mathematics? (3) What constitutes a fundamental math vocabulary that provides students the opportunity to understand college mathematics? We think these questions are worthy of doctoral research.

We are not presuming that improving students’ vocabulary will solve all of the problems for students learning higher mathematics, but we do think this is a
basic part of the problem. A colleague, Claran Einfelt, while working on the Illinois Standards Achievement Test, had an elementary teacher argue that it was unreasonable for the 3rd grade ISAT test to include words such as vertex (instead of corner) because vertex was too hard for students to remember. Einfelt’s response was that 3rd graders have no trouble remembering words such as triceratops and tyrannosaurus rex and can even explain the difference between them. If teachers want to use everyday conversation in teaching, we recommend that they use/speak the correct mathematical term first, and then the vernacular, so that students will associate the correct term with ordinary usage or jargon. For example, the teacher should say “numerator or top number.” Students can and will use the correct math terminology if their teachers use correct mathematical terminology in their daily instruction.

“Learning the language shows respect for the discipline.” Carol Castellon, 2009

Guidelines for Submitting Manuscripts

Submission of manuscript by email is strongly encouraged. Please send an electronic copy of your submission to tvoepel@siue.edu and a reply will be sent when it has been received. When sending an article by email, please include all identifying information in the email, not in the manuscript.

Otherwise, prospective authors should send:

- **Five (5) copies of your article**, typed, double-spaced, 1-inch margins. Put your name and address only in the cover letter. No identifying information should be contained in copies of the manuscript. Articles should be no more than ten pages in length, including any graphics or supplementary materials.

- **A diskette with your article, including any graphics**. We prefer that the article be written in Microsoft Word and that it be saved on an IBM-compatible disk. Graphics should be computer-generated or drawn in black ink and fit on an 8 1/2"x11" page.

- **Your name, address, phone, and e-mail** (if available) should be included in a cover letter.

- **A photo of yourself (Illinois authors only)**, color or black/white.

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Chips, Chopsticks, and Polygonal Numbers to Foster Algebraic Thinking

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In these activities, students represent triangular, square, pentagonal and hexagonal numbers with chips. Chopsticks are used to break polygonal numbers into components. By representing each part with an algebraic expression, students can find an algebraic representation for the total, and establish relations between different algebraic expressions. In this article we focus on breaking pentagonal and hexagonal numbers into triangular numbers. Of course other decompositions of polygonal numbers are possible and are hinted at the end of the article. The goals of these activities are to help students

- find number patterns that have geometrical structure;
- develop their own strategies to count numbers that form a pattern;
- compare their strategies and learn from each other; and
- experience the beauty of mathematics.

The activities were conducted with five groups in a public magnet school in Korea. The students are among the top 20% of 11th grade students and have special talent for English and preference for social studies, but they are not especially gifted in mathematics. For the first activity the teacher illustrated how to represent the triangular numbers algebraically after using chopsticks to partition geometrically arranged chips and finding the numerical representation of the 2nd, 3rd, and 4th triangular numbers. Then the teacher explained a strategy of representing square numbers as the sum of two triangular numbers using chopsticks. Then, after the teacher explained how to form pentagonal numbers with chips, she asked her students to find their own strategies for using chopsticks to partition them and finding numerical and algebraic representations of the pentagonal numbers. Then she asked students to do the same for central hexagonal numbers. The teacher prepared a PowerPoint presentation and asked students to draw imaginary chopsticks on the board where the slides were projected (Figure 1).

![Figure 1. Student expressing a pentagonal number as \((n - 1)xn + \frac{n(n + 1)}{2}\)](image)

Students first wrote the numeric representation of 2nd, 3rd, and in some cases 4th polygonal numbers, then wrote an algebraic representation. Students enjoyed finding many different strategies for just one question and even became competitive. In one class students found eight different ways to represent pentagonal numbers. In another class students were eager to share their strategies even after the bell had rung. Some of the students who did not show much interest in the beginning, after watching their peers present their novel ideas, became interested and tried to come up with their
own ideas. In some cases students were amazed by the different strategies of their peers. Although students were not particularly fond of the chopsticks and wanted to use tools that could be bent like wire, thinking about where to put the chopsticks seemed to help them to concentrate on how to partition the array of chips. Using tools to partition the arrays was useful in finding patterns.

Activity 1: Triangular Numbers

1. The sums of consecutive whole numbers like 1, 1 + 2, 1 + 2 + 3, ..., can be represented as triangular shapes (Figure 2). Represent the first four triangular numbers with poker chips.

\[
T_1 = \frac{1 \times 2}{2},
\]
\[
T_2 = \frac{1 + 2}{2} = \frac{2 \times 3}{2},
\]
\[
T_3 = \frac{1 + 2 + 3}{2} = \frac{3 \times 4}{2},
\]
\[
T_4 = \frac{1 + 2 + 3 + 4}{2} = \frac{4 \times 5}{2},
\]
\[
T_n = \frac{1 + 2 + 3 + 4 + \ldots + n}{2} = \frac{n \times (n + 1)}{2}
\]

Activity 2: Square Numbers

The arrays in figure 5 are called square numbers. The number of chips in each row is equal to the number of rows. So, if there are \( n \) rows, the total amount of chips will be \( n \times n = n^2 \). Use chips to build the first four square numbers.

Find triangular numbers within the square numbers and use a chopstick to separate them (Figure 6).

Express each square number algebraically as the sum of two terms that represent triangular numbers. For example,

\[
S_3 = 3 \times 3 = \frac{2 \times 3}{2} + \frac{3 \times 4}{2},
\]
\[
S_4 = 4 \times 4 = \frac{3 \times 4}{2} + \frac{4 \times 5}{2}.
\]

Generalize to the \( n^\text{th} \) square number. Verify that the algebraic expressions on both sides of the following identity are indeed
equivalent. Expand and simplify the right side.
\[
\frac{n \times n}{2} = \frac{(n-1) \times n + n \times (n+1)}{2}
\]

Activity 3: Pentagonal Numbers

The arrays in figure 7 represent pentagonal numbers. Each new pentagonal number is formed by adding a new layer consisting of three sides at the bottom, thus extending the two sides that meet at the upper vertex. This upper vertex has thus a different role than the other vertices in this kind of pentagonal array.

Pentagonal numbers can be decomposed into triangular numbers. Figure 8 shows one kind of decomposition.

Express each of the pentagonal number as a sum of triangular numbers. For example,
\[
P_2 = 1 + 1 + 3 = 2 \times \frac{1 \times 2}{2} + 2 \times 3
\]
\[
P_3 = 3 + 3 + 6 = 2 \times \frac{2 \times 3}{2} + 3 \times 4
\]

Generalize the pattern to obtain a formula for the number of chips in the \(n\)th pentagonal number.
\[
P_n = 2T_{n-1} + T_n = 2 \times \frac{(n-1) \times n}{2} + n \times (n+1) = n \times (3n-1)
\]

Students can also break a polygonal number into a triangular number and a square number (Figure 10). In Figure 11, the pentagonal numbers have been squeezed to show more clearly how they are formed by a triangular number and a square number (Meavilla Seguí 2005).

Express the \(n\)th pentagonal number as the sum of the \((n-1)^{\text{th}}\) triangular number and the \(n\)th square number. Verify that this algebraic expression is equivalent to the one obtained above.

Students found several partitions and the corresponding algebraic expressions. Verify that indeed the total number of chips is the same. For the partition in Figure 7 students found this expression
\[
1 + (3 \times 2 - 2) + (3 \times 3 - 2) + (3 \times 4 - 2) + \ldots = \sum_{k=1}^{n} (3k - 2)
\]
Figure 13. $\frac{n(n+1)}{2} + (n-1)n$

Students used "wire" for the following partition.

Activity 4: Hexagonal Numbers

The arrays in Figure 15 are called the central hexagonal numbers.

Figure 14. $n + (n+1) + (n+2) + \cdots + (2n-1)$

Use chopsticks to break each hexagonal number into triangular numbers. Several solutions are possible. Figure 16 shows one.

Express the hexagonal numbers in terms of the triangular numbers.

$H_3 = 1 + 6 \times \frac{2 \times 3}{2}$

$H_4 = 1 + 6 \times \frac{3 \times 4}{2}$

Write an expression for the number of chips in the fifth central hexagonal number.

Generalize to the $n^{th}$ central hexagonal numbers, $H_n = 1 + 6 \times \frac{(n-1) \times n}{2}$.

Use the partition of $H_4$ illustrated in Figure 17 to suggest another algebraic expression. Express first this particular hexagonal number as the sum of particular triangular numbers and then generalize. Verify that the new general expression is equivalent to the one obtained above.

Below are other partitions found by students and the corresponding algebraic expressions.

Express first this particular hexagonal number as the sum of particular triangular numbers and then generalize. Verify that the new general expression is equivalent to the one obtained above.
Figure 21 illustrates the fourth central hexagonal number as the sum of hexagonal shells. Students found an expression for the $n^{th}$ central hexagonal number as the sum of the shells $1 + (6 \times 1) + (6 \times 2) + (6 \times 3) + \cdots = 1 + 6 \cdot \sum_{k=1}^{n-1} k$.

Verify that this expression is equivalent to the expressions obtained above.

Final Comments

Polygonal numbers are one kind of geometrical representation of numbers and relations among numbers. Students can use such geometrical representations as a means to explore algebraic ideas. With the help of these representations students can think about the relations among the numbers, express them using their own words, and represent them with letters. These activities can stimulate students to try to find various ways of solving a problem and appreciate the joy of finding various solutions. The activities also foster them to think how to find patterns, to express the patterns in numerical forms, and to generalize them into algebraic forms. A teacher can use geometrical representations to help students as they learn to use algebra to generalize and justify (Flores 2002). Polygonal numbers and other geometrical representations can provide a more concrete step towards the more abstract use of letters as variables or generalized numbers, which for beginners may be a little complicated.

References


CALCULATING TUMOR REDUCTION IN CANCER PATIENTS
A Project for your PreCalculus and Calculus Students

Denise Reid, Vickie Graham, & Janice Lowe – Valdosta State University, dtreid@valdosta.edu

One goal that teachers strive to achieve is to make the presented material more relevant to the students. Often students view mathematics as a set of rules that are applied to solve equations and manipulate variables. The beauty of mathematics lies in the theory and applications to the real world setting. Developing an appreciation and an understanding for the subject and its intricacies can sometimes be a daunting task for a busy teacher. One must be alert to events on the social, political, and educational scene that can be inserted into the lessons to demonstrate the importance and applications of mathematics in our society. In this activity we have chosen an illustration of enhancing one objective from a calculus class within the field of medicine. The activity can also be adapted for use in a precalculus class.

Cancer is a disease that impacts many families. As part of the preparation for the article, we toured our local cancer center and obtained information from the clinical research nurse and the clinical coordinator of the center. When a patient is diagnosed as having a cancerous tumor, a program of treatment is developed. Whether the tumor will be treated with radiation therapy or chemotherapy is dependent upon the type of tumor and the statistics that show the technique with the highest success rate. Initial measurements are established with a CT scan and recorded as a baseline measurement. Then, a program of treatment is developed for the patient. Treatments are normally administered on 6 to 8 week intervals. Follow-up CT scan measurements are made after each interval of treatment for those patients participating in a study. Otherwise, follow-up scans are ordered at the doctor’s discretion. After each measurement, the area of the tumor is compared with the baseline area, and the percent reduction is calculated. The tumor is labeled as PR for partial response, which is at least a 30% reduction, SD for stable disease, or PROG for progressive disease.

Most are approximately elliptical in shape. The CT x-rays show a view of the tumor from side-to-side and front-to-back, but not top-to-bottom. Therefore, the depth of the tumor was not included in the calculations. Occasionally, a tumor is irregularly shaped and the only measurement used is the length of the longest diameter.

The Project

This activity will use the baseline and follow-up measurements of a cancerous tumor to calculate the percent reduction in size of the tumor as the patient undergoes therapy. A teacher might begin the project with a tour of the local cancer center as a class field trip. The oncologist and nurses can give the students a close-up view of the instruments used to measure tumors and methods for recording data. If a visit to a facility is not possible, then an oncologist can be invited as a guest speaker for a presentation to the class.

Area of an Ellipse Derivation:
Consider an ellipse centered at the origin with major axis segment $V_1V_2$ and minor...
axis segment \(B_1B_2\). The equation of the ellipse is 
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]
Solving the equation for \(y\) yields 
\[
y = \pm b \sqrt{1 - \frac{x^2}{a^2}}.
\]
Renaming this function we have 
\[
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,
\]
To determine a formula for the area of the ellipse, partition the interval from \(V_1\) to \(V_2\) into \(n\) subintervals and examine the \(i^{th}\) subinterval. The rectangle used to approximate the area of the \(i^{th}\) subinterval has base \(\Delta x_i\) and height 
\[
f(x_i) = \pm b \sqrt{1 - \frac{x_i^2}{a^2}}.
\]
An estimate for the area of the region formed by the \(i^{th}\) subinterval is 
\[
\Delta A_i = \int_{x_{i-1}}^{x_i} \pm b \sqrt{1 - \frac{x^2}{a^2}} \, dx.
\]
By summing the areas of all \(n\) subintervals, we get an estimate for the area of the ellipse.
\[
A \approx \sum_{i=1}^{n} \Delta A_i(x) = \sum_{i=1}^{n} 2 f(x_i) \Delta x_i
\]
Therefore, 
\[
A = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} 2 f(x_i) \Delta x_i = \int_{-a}^{a} 2 f(x) \, dx
\]
Using trigonometric substitution, let 
\[
x = a \sin \theta.
\]
Thus, 
\[
x^2 = a^2 \sin^2 \theta \quad \text{and} \quad dx = a \cos \theta \, d\theta.
\]
The indefinite integral needed can be found as follows:
\[
2 \int_{-a}^{a} 2 b \sqrt{1 - \frac{x^2}{a^2}} \, dx = 2ab \left(\sin^{-1} \frac{x}{a} + \frac{x \sqrt{a^2 - x^2}}{a}\right)
\]
\[
= \pi ab
\]
\[
\therefore \text{The area of an ellipse with length of major axis } 2a \text{ and length of minor axis } 2b \text{ is } \pi ab.
\]
Example:
The equation for an ellipse centered at the origin is 
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
where the length of the
major axis is $2a$ and the length of the minor axis is $2b$. Suppose the baseline measurements are $4cm \times 3.5cm$. Therefore, $a = 2cm$ and $b = 1.75cm$. The area formula for the ellipse, $A = \pi ab = \pi (2)(1.75) = 10.996sq cm$. After the first follow-up scan, the tumor measures $3.6cm$ by $3.12cm$, and the area of the ellipse would be $A = \pi (1.8)(1.56) = 8.822sq cm$. To find the percent reduction, we subtract the new area from the original area, divide by the original area. Then, change your answer to a percent by multiplying by 100. Percent decrease = 
\[
\frac{10.996 - 8.822}{10.996} \times 100 = 19.8%\]. Therefore, the tumor's cross-sectional area has decreased approximately 20% after one treatment.

Conclusion:

We encourage those who use this idea to adapt any part of the project to accommodate your classes and continue asking, “What other extended activities can be used with this information?” We found that visiting the Pearlman Comprehensive Cancer Center at South Georgia Medical Center in Valdosta, Georgia, was most informative. Special thanks to Maryann Heddon, Clinical Research Nurse and Julie Huxford, Clinical Coordinator for providing the data we used in the article and for their assistance.

Your presence is requested at the 60th Annual Meeting and Conference of the Illinois Council of Teachers of Mathematics, “Innovations in Teaching Mathematics”. It will be held October 15-17, 2009 in Peoria, Illinois. We hope to see you there.
Worksheet 1

The first patient had five tumors in the lower lung. The location of the first lesion was the left lower lobe. Complete the table below.

<table>
<thead>
<tr>
<th>Tumor Measurements</th>
<th>Area</th>
<th>Percent Reduction in area</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Baseline Measurements</strong></td>
<td>Date: 10/15/01</td>
<td>3.5 cm x 3.5 cm (a=1.75 and b= 1.75)</td>
</tr>
<tr>
<td><strong>Follow-up #1</strong></td>
<td>Date: 11/20/01</td>
<td>1.9 cm x 3.1 cm (a=.95 and b= )</td>
</tr>
<tr>
<td><strong>Follow-up #2</strong></td>
<td>Date: 01/03/02</td>
<td>2.1 cm x 2.1 cm (a= and b= )</td>
</tr>
<tr>
<td><strong>Follow-up #3</strong></td>
<td>Date: 02/08/02</td>
<td>1.4 cm x 2.1 cm (a= and b= )</td>
</tr>
</tbody>
</table>

Chart the area of each follow-up measurement by drawing a line graph.

Between which consecutive measurements was the decrease the greatest?

Did any of the measurements remain constant? Increase?

Is the therapy effective? How would you label the tumor’s response to treatment (PR, SD, PROG)?

Assuming the tumor continues to follow the same pattern as the last two measurements, write an equation for that linear pattern, and use it to predict the next measurement.
## Worksheet 2

Complete the table below which gives the data for the patient’s second tumor (located in the right lower lobe.)

<table>
<thead>
<tr>
<th></th>
<th>Tumor Measurements</th>
<th>Area</th>
<th>Percent Reduction in area</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Baseline Measurements</strong> Date: 10/15/01</td>
<td>1.0 cm x 1.5 cm (a=      and b=     )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Follow -up #1</strong> Date: 11/20/01</td>
<td>0.5 cm x 1.0 cm (a=     and b=     )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Follow–up #2</strong> Date: 01/03/02</td>
<td>0.5 cm x 0.5 cm (a=     and b=     )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Follow-up #3</strong> Date: 02/08/02</td>
<td>0.5 cm x 0.5 cm (a=     and b=     )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Chart the area of each follow-up measurement by drawing a line graph.

Between which consecutive measurements was the decrease the greatest?

Did any of the measurements remain constant? Increase?

Is the therapy effective? How would you label the tumor’s response to treatment (PR, SD, PROG)?

Assuming the tumor continues to follow the same pattern as the last two measurements, write an equation for that linear pattern, and use it to predict the next measurement.
Enrichment Activities:

1.) Divide the class into groups. Each group will collect data from a local oncologist to track the progress of their own patient. At the end of the treatments, each group will present their data to the rest of the class using charts and graphs.

2.) An extension of the area of an ellipse is to calculate the volume of an ellipsoid with radii a, b, and c. The volume of an ellipsoid is \( V = \pi abc \). Students are divided into groups. Each group is assigned a particular brand of eggs. They are to collect for their brand a small, medium, large, and an extra-large egg. They are to measure the circumference of each egg and use the formula for the circumference of a circle to find the value of the radius of the egg, which gives them the value of a and b. They are to measure the length of the egg and divide it by 2 to find the value of c. Then using the formula for the volume of an ellipsoid, they are to find the volume of each type of egg for their brand, organize the information in a table, and find the percent increase in volume for each size egg. The class then compares brands.
Sometimes an idea hits you when you least expect it! I teach math teachers and teacher candidates and am always telling them that, “Math is everywhere.” Although I look at everything mathematically, I never expected my summer vacation away from school to provide such a perfect hands-on model for use in elementary school. We have all heard about teachable moments but this article is about a teaching moment… what I call a situation that lends itself to use in teaching a particular concept. Here is what happened to me a few summers ago. I was camping in Michigan with a few of my friends. The first night, I was alone because my friend had called to say that she would not be coming until the next day. So I went down to the lake, put a worm on my hook and began to fish. I thought about what might happen if I did not catch anything. The next day, my friend and I went fishing again. My friend, who is a first grade teacher, and I talked about what we might do if we had no food and could not catch any fish to eat. Thus began the idea of connecting worms, first grade and fractions. Although fractions are not usually studied in detail in first grade other than to “understand and represent commonly used fractions” (PSSM, p. 78) and “the development of rational-number concepts is a major goal in grades 3-5” (PSSM, 2000, p. 35), much can be done with fraction concepts in first grade when they are linked to something fun and relevant to the children’s real world. There are several different ways to interpret fractions…part-whole meaning, quotient meaning, ratio meaning and operator meaning. (Kieran, 1980) The notion of fractions as whole number division (quotient) is not the usual method for introducing fractions as, typically, it is the part-whole interpretation that is used to introduce them. But, first graders have an intuitive idea of sharing and, in particular, sharing fairly, as when they are asked to distribute supplies or goodies to each other and are told to be fair. Since this “fair-share” concept is really one interpretation of division, the idea of division can be used by first graders without the formal introduction to it as a mathematical operation. This whole number division interpretation of fractions also leads to two different ways to interpret the fraction 3/4…a part of a whole that is equally shared among 4 groups and 3 of those parts combined … or 3 wholes equally shared among 4 groups. Once first graders understand the quotient model, they can not only represent fractions, but can begin to compare them and to understand some simple addition, subtraction, multiplication and division properties.

The following “script” is typical of a lesson I have presented to my friend’s first grade class at least 4 times. It begins with a story that is “partially” true that involves their classroom teacher and the lesson presenter. The bold print will represent my comments and the comments in bold italics will be typical student comments.
The classroom teacher, Jan (this is actually her real name), introduces me as her friend, and says that I have been listening to their reading lesson and I want to share my poem with them. The students gather on the floor in a semicircle around me. I have a bucket in one hand but they cannot see what is in it.

My name is Nan and your teacher’s first name is Jan which rhymes with Nan. I have a poem to read to you. See if you can figure out what comes next.

There once was a woman named Nan
Who liked to go camping with Jan.
She paid no rent
‘Cause she slept in a tent
And loaded her things in a van.

The first day Nan waited for Jan
And hoped she’d bring food in a can.
Nan wanted some meat
Or anything to eat.
Just something to cook in the pan.

But Jan never came in her car.
Nan had already eaten her candy bar.
She decided to fish
For something to put in her dish,
For the store was much too far.

Everywhere she did look.
This a whole hour took.
While digging around
At last she found
A worm to put on her hook.

She dangled the worm in the pond.
And she sat, not making a sound
When she pulled up the line
Expecting a fish to find
Only the worm could be found

The day had turned into night
She had not even eaten a bite
What was she to do?

She needed some food
There was nothing to eat in sight

EXCEPT……

Your teacher and I are good friends and every year we really do go camping. What do you know about camping?” Answers usually include comments like… You sleep in a sleeping bag. You stay in a tent. You swim in a lake. You ride in a boat. You sit around a fire. Sometimes a prompt is needed to get the response you eat outside.

Sometimes you have to find your own food when you camp. I got to the camp site first, set up the tent, started a fire, and waited for Jan to get there. My cell phone rang and it was Jan telling me that she would not be coming until the next day because she had to finish putting up the bulletin board in her classroom. After I hung up, I realized that I had been waiting for Jan to bring the food. I had nothing to eat. Where can you find food when you are camping? Responses usually include Go to the store (sorry…no store close enough), Catch a rabbit (sorry…no rabbits in sight). Sometimes a prompt is needed to get the response, you go fishing. The only thing is I didn’t have any bait to catch the fish. What could I use? One of the students always shares that you use worms to catch fish.

That is exactly what I did. I found a worm, put it on the end of a string and held it over the water. It was 5:00 at night and I was really hungry. I pulled the string out of the water and…no fish. I put it back in, waited until 6:00, and pulled it out of the water and…no fish. I did the same thing at 7:00 and 8:00. Now it was dark, I was really hungry and I had no fish. What do you think I did? …I ate the worm. This usually elicits responses like yuck or no way.
Guess what’s in my bucket? ... Worms. I pull one out and wiggle a gummy worm in front of them. Let’s go to our seats and we will talk more about worms.

On each desk their teacher has placed two paper plates. Put one of the paper plates to the side. I pass out one worm to each student. Put the worm on the top part of the plate, horizontally (sideways), and write the number “1” next to it, on the left (because you have one worm). I show this on the ELMO:

(A nice thing about gummy worms is that they stick to the plate and the teacher can hold the plate up and the worm will not fall off.)

This is what I ate the first night. I ate one whole worm and then I went to sleep.

The next day, your teacher, Jan showed up. I was so glad to see her. I told her about having to eat the worm and I was glad that I would not have to do that again. She said, “I thought you brought the food. I do not have any.” Do you know what we had to do? We had to try to catch a fish again. We found another worm, exactly the same size as the one I had caught the day before. We put it on the string, dangled it into the water and waited. When we pulled it out at 5:00...no fish...at 6:00...no fish...at 7:00 and 8:00...no fish. We had nothing to eat. What could we do? Someone always says you can eat the worm. But there are two of us...how can we both eat it? If I eat the whole worm, there will be none for Jan and if Jan eats the whole worm, there will be none for me. What are we going to do? Someone usually says you will have to share it.

I carefully take a second worm out of the bucket and show that it may be a different color but it is the same size as the worm from day 1. I then split it into 2 pieces but make one piece much larger than the other. I show this on the ELMO.

Great...I will eat this piece (pointing to the larger piece) and Jan will eat this piece (pointing to the smaller piece). Oh no, they say you can’t do that. Why not? Because that is not fair...you both should have the same amount. Oh, the pieces have to be the same size to be fair. (An interesting note is that one time I assigned the larger piece to their teacher, and they thought that was just fine!!!) I now take the bigger piece and break off some so that it is the same size as the smaller one. I hold up the two same-sized pieces and say, these are the same size now. Is this okay?

I show this on the ELMO.

No some say. There is some left over and you should not waste food. Others say there are 3 pieces and only 2 of you. Okay... why don’t you show me how to divide the worm into 2 pieces that are the same size? I give each student another gummy worm, the same size as the first worm, and a plastic knife and they try to cut the worm into 2 equal-sized pieces. How did you do that? Some say, I folded it and cut it. Others say, I counted the ridges and cut it so that there were the same number of ridges on each side (gummy worms have very distinct ridges on them). Still others may say I cut it by the colors (gummy worms are usually 2 colors, with 2 alternating stripes of each color). And how do you know that the pieces are the same size? They put the pieces next to each other on the second paper plate to show me. Those students who may not have the same size pieces can cut off part of the larger piece and stick it to the smaller one. (Another nice thing about
The gummy worms is that they are forgiving and pieces will stick together.

Put the 2 pieces next to each other horizontally under the first worm you have on your plate. Next to it we will write the number 1 (because we had one worm) and put a line under that and write the number 2 below the line (because there are two equal-sized pieces). Notice that the 2 equal sized pieces together are the same length as the whole worm. I show this on the ELMO:

Jan and I went to sleep in our tent. We knew that the next day our other friend, Pam (this really is her name), was coming and she always brings good food. The next morning our friend Pam came and we told her about having no food and that we could not catch any fish. She said, “Oh dear…I thought you had the food, I did not bring any.” What can we do? Catch a worm and go fishing the students say.

We caught another worm, again the exact same size as the other two worms, put it on the string, dangled it into the water and waited. When we pulled it out at 5:00…no fish…at 6:00…no fish…at 7:00 and 8:00…no fish. We had nothing to eat. What could we do? Eat the worm, they say.

I take another worm out of the bucket, fold it in half and cut it and show them only 2 pieces. They usually tell me you do not have enough pieces 'cause there are 3 of you. You have to cut it again. I cut one of the halves into 2 equal sized pieces and show them the 3 pieces. I show this on the ELMO.

I will eat this piece (pointing to the half), Jan will eat one of these pieces (pointing to one of the smaller pieces) and Pam will eat this piece (pointing to the other small piece). Oh no, they say, that is not fair. Why? There are 3 pieces and 3 of us. The pieces are not the same size, your piece is bigger than the others and that is not fair. Why don’t you show me how to be fair? I give them each another worm, and show them that it is the same size as the worm from day 1 and the same size as the two pieces from day 2 put together. They try to cut it into 3 equal sized pieces. How did you do that? Some say I folded it into an “S” shape. Some say I just guessed, and others say I counted the ridges. How do you know that they are all the same size? They place the 3 pieces next to each other on the other paper plate to show that they are the same. Any students who do not have the same sizes can cut a little off the bigger piece and stick it on the smaller one until they are the same. Put the 3 pieces next to each other horizontally under the second worm you have on your plate. Next to it we will write the number 1 (because we had one worm) and put a line under that and write the number 3 below the line (because there are three equal-sized pieces). I show this on the ELMO:

Guess what happened the next day? Your principal, Mrs. X (I use the real name of the principal) joined us. (If the principal is male, however, you may not want to use him so that the discussion does not begin about men and women sharing a tent!) How many of us were there now? Four. And what do you think we did? By now they can tell me the story. Mrs. X brought no food; you had to catch a worm. It was the same size as the worms you had found on day 1, 2 and 3. You put it into the water but caught no fish.
and you had to share the worm between the 4 of you.

I give them another worm; they cut it into 4 equal-sized pieces. **Put the 4 pieces next to each other horizontally under the third worm you have on your plate.** Next to it we will write the number 1 (because we had one worm) and put a line under that and write the number 4 below the line (because there are four equal-sized pieces). Notice that the 4 pieces of worm put together make the same size as the 3 pieces of worm put together and as the 2 pieces of worm put together and the whole worm. I show this on the ELMO: Let’s go back and tell the story from the plate. They retell the story pointing to the pieces on the plate and saying *this is the piece you ate on the first day, second day, third day, and fourth day.* When they get to the part where they are reading the numbers on the side I tell them... **1/2 is called one half** (some students want to call this one second at first because it’s from the second day and they may know this as the ordinal number). **1/3 is called one third** (because it’s from the third day). **1/4 is called one fourth** (because it’s from the fourth day).

Now I ask them the following questions and use the whiteboard to write their answers. They can look at the pieces from the plate to find out the answers and some of them will need to pick up the pieces and place them on the other plate to check. **What is bigger...one half or one third?** 1/2. I write and say 1/2 is bigger than 1/3 (1/2 > 1/3 if they have had this notation) on the whiteboard and show this on the ELMO: Let’s talk about what we are seeing here. When the number of people is 2 the pieces are bigger than when the number of people is 3...is that right? And so which is bigger...one third or one fourth? 1/3. I write and say 1/3 is bigger than 1/4 (1/3 > 1/4) and show it on the ELMO. So that also means that when the number below the line is smaller, the pieces are bigger...is that right? And that is why 1/2 which has a 2 on the bottom and a 1 on the top and means that 2 people shared 1 worm is larger than 1/3 which has a 3 on the bottom and a 1 on the top and means that 3 people shared 1 worm. And it also means that 1/3 which has a 3 on the bottom and a 1 on the top and means that 3 people shared 1 worm is larger than 1/4 which has a 4 on the bottom and a 1 on the top and means that 4 people shared 1 worm. Is that right?

Now let’s take pieces off of the plate one at a time…. in this order 1/2, 1/3, 1/4, 1/3, 1/4, 1/4, 1/2., 1/4. (When it is acceptable to the classroom teacher, the students can eat the pieces as they take them off the plate.) Then I ask what piece is left? This is a mini assessment for me and their teacher to check because if they did it correctly, they should have one 1/3 piece left.

I now pass out the following paper to each student and display it on the ELMO. I tell them write “1” above the line and 1 below the line next to the first worm and color all of it. This is because it is 1 worm for 1 person. Write “1” above the line and 2 below the line next to the second worm, divide it into 2 equal-sized pieces and color one of them. This is because it is 1 worm for 2 people. Write
“1” above the line and 3 below the line next to the third worm and divide it into 3 equal-sized pieces and color one of them. This is because it is 1 worm for 3 people. Write “1” above the line and 4 below the line next to the fourth worm and divide it into 4 equal-sized pieces and color one of them. This is because it is 1 worm for 4 people. I ask them, See if you can figure out what would happen on a fifth day and what would they write next to the bottom worm and color one of the pieces. This would be 1 worm for 5 people. (This is for some of the higher level thinkers and usually several can do this correctly.)

This is what their worksheet should look like when completed.

Then I ask them, do you notice anything about the worms as we go from top to bottom? The worms are all the same size. And do you notice anything about the face of the worms as they go from the top to the bottom? If no one notices I prompt them to look at the mouth and usually someone notices the mouth is going from a smile to a frown. I tell them, this is because the piece on top you have colored is the biggest and as you go down the sheet, the pieces get smaller. The worm is happy when what is colored is biggest and sad when what is colored is smallest. This is an additional assessment that the students can take home. Now I will finish the poem.

That night Nan had her meal
This is what happened...for real
Even though it did squirm
She ate the whole worm
So much better did she feel.

The next day she had to share
For now they were a pair
One half for Jan
One half for Nan
Of course, she had to be fair

The next day, what would it be?
Jan, Pam, and Nan make three
Each got a third
She ate like a bird
The piece of the worm was tiny

And then the group was four
To cut the worm made their hands quite sore
Each piece was so small
It was like nothing at all
Each got a fourth, not a bit more

When camping and there’s no food
There’s something you must do
Your worm you must share
And the pieces must be fair
More people, less worm for you.

The gummy worm as a concrete model for teaching fractions is very powerful. “Fractions have always represented a considerable challenge for students even into the middle grades.” (VanDeWalle 5th edition, p. 242) and NAEP testing has always shown that students do not understand fraction concepts. (Wearne & Kouba, 2001). The gummy worm manipulative allows teachers and students in early grades to discuss, compare, add, subtract, multiply, and divide very simple fractions using the quotient or whole number division interpretation. Worms can also be easily integrated with literature in the form of a poem and a “fantasy” tale, lead to some paper plate art work and satisfy hunger.

I have also extended this lesson in 2nd grade classes to talk about the case where we catch 3 worms and have to share them
among 2 people. Some students will give each person 1 worm and split the other in half so that each person gets 1 and 1/2 worms and other students will split each worm in half and give each person 3 pieces so that each person gets 3 halves. Who would believe that 2nd graders can show that $1 \frac{1}{2} = \frac{3}{2}$? In one 3rd grade class we actually talked about $1/2 + 1/3$...but that discussion may be for a future article. And maybe, when the students encounter the algorithms for operating with fractions in 4th grade, the visual model of worms hopefully will come back to them and they won’t have to S-Q-W-O-R-M when they forget a step from the rule!

Bibliography


Your presence is requested at the 61st Annual Meeting and Conference of the Illinois Council of Teachers of Mathematics, “Building Communities, Creating Opportunities”. It will be held October 14-16, 2009 in Springfield, Illinois. We hope to see you there.
Japanese Chess (Shogi) is played with pieces that look like the figure below.

A nice name for this figure might be an isosceles pentagon. The isosceles pentagon has never been formally analyzed. Thus, your class can have the opportunity to do some original mathematics. The class can build up a system of definitions, conjectures, and theorems about a shape that has yet to be explored. What follows is a lesson that guides students through geometric organization, while allowing them the freedom to do original investigation.

First we need a definition. There is no correct definition, because this is a totally new figure. So your definition cannot be wrong. However, you need to make sure your definition does the job. Here is a sample definition that is inappropriate.

INSUFFICIENT DEFINITION:
An isosceles pentagon has two pairs of equal sides.

This definition is inappropriate because it doesn’t say that it is a pentagon. It might be referring to a hexagon or an octagon. It is also inappropriate because it includes figures you may not want, as illustrated in the figure below.

Since we already have the terms isosceles and pentagon, there should be some consistency. Let’s back up to isosceles triangle. Here are several good definitions of isosceles triangle.

- An isosceles triangle is a triangle with at least two congruent sides.
- An isosceles triangle is a triangle with at least two congruent angles.
- An isosceles triangle is a triangle with at least one line of symmetry.
- An isosceles triangle is a triangle that is the image of itself in a reflection.

All of these definitions describe exactly the same thing. Any of these definitions can be used without changing geometry as we know it.

The activities on the following pages can be used in a classroom.
Isosceles Pentagon Activities

Part 1
1. Circle the figures below that you want to consider to be isosceles pentagons. Remember, there is no right answer.

2. Write a definition of isosceles pentagon that includes all figures you circled, but excludes the ones you didn’t circle.
3. How would you construct an isosceles pentagon on a dynamic geometry system like Geometer’s Sketchpad or Cabri?

Part 2
An isosceles triangle has a vocabulary associated with it. An isosceles pentagon will probably need a larger vocabulary. You might wish to include terms like base, vertex, and so forth, but you will need more terms. Each part with question marks should be named. Remember, this figure has never been studied before, so there are no standard names. Be original. Be creative.

1. In the chart below, define the terms that are given for an isosceles pentagon. Cross out the ones you don’t wish to use. Add at least three other terms and their definitions.

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
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<tbody>
<tr>
<td>Vertex</td>
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<tr>
<td>Base</td>
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<tr>
<td>Base Angles</td>
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</tbody>
</table>

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2. An isosceles pentagon has special types of diagonals. Let’s give these diagonals names. Put your names for the diagonals in this chart and provide a definition.

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
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Part 3

A definition identifies a figure, but more things will be true about the figure than are found in the definition. For example:

Definition: A parallelogram is a quadrilateral with two pairs of parallel sides.

On examining parallelograms, other things seem to be true:
- Opposite sides of a parallelogram are equal.
- Opposite angles of a parallelogram are equal.
- A diagonal divides a parallelogram into two congruent triangles.
- Diagonals of a parallelogram bisect each other.
- The area of a parallelogram is its base times its height.

Each of these is a conjecture. When it is proved, it is called a theorem.

1. Make (at least) five conjectures about an isosceles pentagon.
2. Choose (at least) two of your conjectures and prove them.

Examples

I hate to give examples, because examples encourage imitation, not creativity. But here is a snippet from work submitted to me by Brendan Bond, Shannon McDowell and Lena Sellen, students from Aurora Central High School in Aurora, Illinois.

**Proof:** Arms of an isosceles pentagon are equal.

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Teaching Scope and Sequence Using Concept Maps

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How do you get pre-service teachers to realize that the textbook may not always have the best layout for their students’ needs? Mathematics methods instructors know that teacher candidates’ views of scope and sequence are often limited to the textbook. In an effort to help teacher candidates realize that textbook authors may not know what is best for the students in their classroom, I introduced concept maps in my Secondary Mathematics Methods course, which consisted of juniors and seniors. The goal for the teacher candidates was for them to gain a better understanding of scope and sequence and to open their minds to deciding what is best for their students.

The Assignment Part I

The importance of curriculum was discussed earlier in the course. The discussion of concept maps included sharing math and non-math examples, as well as different forms, such as spider, hierarchy, and flow chart. Finally, it was time to give the students an opportunity to create their own concept maps. The students selected a small piece of paper with a general mathematics topic written on it. The topics were polygons, functions, equations, and polynomials. As this class is for both middle and secondary pre-service mathematics teachers, a range of topics was a necessity. The written assignment distributed to the students consisted of the following.

The purpose of the concept map in this context is to look at how the pieces of a particular topic ‘fit together.’ You are beginning to have a good feel of how the mathematics topics are interrelated. The assignment is for you to create a concept map on your designated topic. You may have to design the concept map more than once. The first draft may be more of a brainstorming session while the eventual final draft will have all the connections identified with labels.

In addition to information regarding when the concept map was due (the next class meeting), the assignment sheet contained the following:

You need to bring two copies to class. One copy is to turn in to me, and the other copy will be used in class. Do not collaborate or work with others on this assignment. I want your view of the topic. There is not a right or wrong concept map, as it is your conceptual view of the topic at hand.

In addition to including it in the assignment sheet, the importance of not communicating with class members about their concept maps was stressed as each individual’s view was to be represented in the map. Additionally, what was not shared with the students was that some of them had the same topics, and they would be coming together to discuss their mutual views on the topic in the next class meeting.

At the next class, there was a range of the formats of concept maps. Some students used the draw feature in Microsoft Word, others drew the maps by hand including color coding, and still others found free software on the internet to download and use. There were differing
levels of detail; some students included very broad categories for the nodes, while other students included numerous details including labeling the connectors, which had been encouraged. The concept maps were collected with the only comments made being about the differing formats. The students had already shared the information about difficulties they encountered drawing with Microsoft Word, as well as the successes experienced using Microsoft Excel and software from the internet. They were not allowed to discuss the content in class as this was part of the next step.

The Assignment Part II

The next phase of the assignment had several motivations. First, the students’ realizations that they may have different views from their classmates as to what subtopics were most important or relevant to their main topics. Additionally, if they did have differing opinions, they needed to share them and negotiate for what should be included and/or omitted, just like they may have to do in the future with their colleagues. Secondly, along that same line of thought, the students came up with their view of the sequencing of topics and needed to give a justification and negotiate when there were disagreements. Finally, textbooks were introduced into the situation as a comparison after students had come up with their sequencing for them to see an author and/or publisher’s view of the topic. The written assignment distributed to the students consisted of the following.

Part I: Find your classmates that had the same topic as you. Share your concept maps with each other. In your group, come to a consensus as to what are the most important pieces of the topic. Create a new concept map as a group.

Part II: Using your concept map, select the order in which you believe the subtopics should be taught. Attach this to the group’s concept map. Justify your ordering.

Part III: Using the textbook assigned to your group, select the sections in the text that align with subtopics. Record these sections. Finally, estimate the length of time (class periods/days) you need to spend on each of the topics.

The students worked in pairs or groups of three. In most cases, they took one of the existing concept maps (usually the one considered by the group to be done the neatest) and worked from it. The students talked at length about their views of the topics, including what was essential to an overall understanding of the topic, subtopics that some students included that others omitted, and the importance of ordering for the subtopics. The students struggled with the amount of detail to include on the concept map, as well as describing the subtopics. The discussions went well beyond the surface level, which became abundantly clear once the textbooks entered the picture.

The textbooks shared with the groups were teacher editions since they included suggested length of time on the topics/lessons. The students in this course are in their semester prior to student teaching or are juniors currently completing their secondary block internship. The students relied heavily on their experiences in the classroom in supporting their ideas for the sequencing, especially the allotted time. In many cases, they did not agree with the textbook’s suggested allotted time. In some cases, I silently agreed with them. In other instances, I opted to let them learn from experience. I had numerous discussions.
with each of the groups as they struggled with the idea of how they could teach the appropriate content for understanding with such a limited time schedule in an academic year. I could not resist saying to them, “Welcome to teaching. Now you understand a little bit of what classroom teachers struggle with everyday.”

As the end of class neared, the class came together for a short discussion. Students shared their difficulties in planning the scope and sequence for such broad topics. The students started recognizing the interrelatedness of the topics throughout the middle and high school curriculum. Several of the students indicated a feeling of being overwhelmed when considering they were only trying to plan for one major topic, but in the future they will have so many more for each class that they teach. The students worked together outside of class to complete their concept map assignment.

Student Work

Most of the work submitted by the students indicated they had spent a significant amount of time thinking about what was important for students to understand the given topic. As the students indicated, they had trouble knowing how much to include when initially creating their concept maps individually since the assigned topics were so broad. For instance, one of the students who was assigned equations as her topic defines an equation in simplistic terms as if for a basic algebra course. However, she also includes logarithmic, exponential, and trigonometric equations as nodes (see fig. 1.), which goes well beyond the scope of a basic algebra course. Additionally, she labels all of the connectors so that branches of the concept map may be read nearly as a sentence.

In figure two, three students collaborated on a concept map for functions. It is interesting to note that the final product from this group was a true coming together of the three versions created by the students. They negotiated what should be included and talked longer than any of the other groups. The one piece of the concept map that they omitted that had been on one of the individual’s concept maps was the labeling of the connectors. I am unsure if this was a purposeful decision. Perhaps it was simply the challenge of putting the labels on the connectors as the student in the group that had included labels on her individual concept map had used Microsoft Word, but then wrote in the connecting words by hand. The students did use different shapes for the nodes depending on where they were on the branches, specifically differentiating between topics and subtopics.

Student Reaction

Although the students found it a challenge to decide what and how much should be included in their concept maps, they gained a great appreciation for the power of the concept map. One of the most important lessons they learned, alluded to earlier, is that teachers do so much more than just open up a textbook and teach. The students quickly acknowledged the difficulties they had determining how inclusive they should be with their topics, and then in their planning, they found it quite challenging to determine an adequate amount of time on the topic, keeping in mind that they would have numerous other topics to teach as well.

As the goal for using the concept map was for the students to gain a better understanding of scope and sequence, which I believe they did, an even better outcome was achieved. One of the students
suggested using the concept map as a form of pre-assessment on a mathematics topic to get a feel for what the students already know or misunderstand about a topic. This suggestion started a whirlwind discussion among the class of how they can incorporate concept maps into their classes to further assist in the development of their students’ conceptual understanding.

Conclusions

With the positive reaction of the students to the concept maps, their excitement to include concept maps in their future classes, and the seriousness with which they took the assignment and produced such good products, I will certainly use this assignment again. After careful reflection on this assignment, I have decided that although there is always room for improvement, I do not want to change the format of this assignment because of the collaborative aspect of it. However, I do think I will include concept maps in another way in addition to this assignment.

The students teach several mini-lessons throughout the course with a major lesson presentation near the completion of the semester. The students are allowed to select the topic for their major lesson presentation from a broad list of middle and high school mathematics topics. In the future, I may have the students complete a concept map on the selected lesson topic so that they can get a better feel of all the underlying mathematical concepts involved in their lessons. Sometimes teacher candidates at this stage do not realize how much mathematics is involved in each lesson as they can only see the small picture, and thus, they gloss over too many of the important details of the lesson because of assumptions they make about students’ prior knowledge. A concept map may help to reveal to them the many different layers to the lesson.

With the need to meet state standards and align curriculum to the standards and identify benchmarks, this is a great exercise. In addition, the students gain a valuable experience in teamwork. This will only benefit them in their future endeavors. Finally, I hope that other teachers of mathematics methods courses will incorporate this assignment into their courses to help teacher candidates gain a better understanding of scope and sequence.

Author note: Thank you to Hannah Combs, Amy Ferries, Tai Horton, and Jonelle Reynolds for the use of their concept maps in this article.

Sample Concept Maps are on the next page.
Figure 1: Student’s Concept Map of Equations
Figure 2: Student Group’s Collaborative Concept Map of Functions
Quick Calculation: Skills to Improve Number Sense in Number Operations

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Abstract

This paper features several strategies that will enable students to make lightning calculations in their heads. The methods will increase students’ confidence in doing mental mathematics.

Quick Calculation: Skills to Master Number Sense in Number Operations

Every day people around the world use mathematical concepts and techniques in the marketplace to conduct commercial transactions. Sticker (1945) stated that “Arithmetic is a science, but calculation is an art. Science is knowledge, art is skill. You have all the knowledge you could possibly need to determine that 57 times 25 equals 1425, but if you are asked to multiply 57 by 25 and cannot do this mentally in just about one second, you are not adept at the art of calculation” (p. 1).

We get used to counting up when a cashier gives us change. For example, suppose you pay $100.00 to buy $87.40 in groceries. The cashier will give you one dime back and count $87.50; next the cashier will give you two quarters and say, “$87.75, $88.00.” Third, you will get two dollars and count $89.00, $90.00. Finally, you will get $10.00 and count $100.00. The process of this traditional calculation takes many steps to finish the transaction. Is there an efficient way to do addition and subtraction? How can one do a quick calculation? How can this skill be developed?

Essentially, it can be done by use of number sense. An ability to sense and visualize relationships among numbers is important. It includes calculation skills with numbers as well as a sense of number and operation and an ability to appropriately use estimation and mental math in computation. For example, when students use the commutative property, they know that $20 + 10$ is the same as $10 + 20$.

The National Council of Teachers of Mathematics in its Principles and Standards for School Mathematics (2000) describes number sense as the ability to do the following:

- Understand numbers, ways of representing numbers, relationships among numbers, and number systems.
- Understand meanings of operations and how they relate to one another.
- Compute fluently and make reasonable estimates (NCTM, 2000, p. 78).

The purpose of this paper is to introduce several methods that will enable students to make calculation in their heads. They will increase their skill in finding relationships among numbers and then compute fluently. Teaching these methods will help students to excel at mathematics calculation (Handley, 2003). In addition, students will increase their confidence in learning mathematics.
Methods for Quick Calculation

Method 1: Addition-Round Up to the Next Power of 10

If one addend is close to a power of 10, round it to that power. Next, add the two numbers. Then subtract the difference between the power of 10 and the first addend. For example, 9996 + 1234 = ?

First, round 9996 to 10000. Next, add 10000 and 1234. The sum is 11234. Finally, subtract 4, the difference between 10000 and 9996. The answer is 11230 as follows:

Fig. 1. Addition-Round Up to the Next Power of 10

Method 2: Addition-Round Up to the Next 10

If all addends are close to the same multiple of 10, round them to that number and add. Then, find the difference between the multiple of 10 and each original number and add the differences. For example, 58 + 57 + 55 + 56 = ?

First, round 58 to 60, 57 to 60, 55 to 60 and 56 to 60. Add all addends and get 240. Find the difference between 60 and 58, 60 and 57, 60 and 55, 60 and 56 and add the differences. The sum is 14. Third, we subtract the sum of the differences which is 14, from the sum of the rounded addends, which is 240. The answer is 226 as follows:

Fig. 2. Addition-Round Up to the Next 10

Method 3: Addition – Add Compatible Numbers – Make Tens in Pairs

Find pairs of addends whose sums are 10. Add the 10s. Add the remaining addends. For example, 9 + 8 + 6 + 7 + 2 + 4 + 3 = ?

First, we add 8 and 2, 6 and 4, 7 and 3 in pairs, respectively. Make tens in pairs. 10 + 10 + 10 = 30. Finally add 30 and 9 to get 39 as follows:

Fig. 3. Addition – Add Compatible Numbers - Make Tens in Pairs

Method 4: Addition – Add Compatible Numbers to Make Tens in Groups

Find groups of addends whose sums are 10. Add the 10s. Add any remaining addends. For example, 2 + 7 + 1 + 4 + 3 + 3 + 2 = ?

First we make tens in groups, i.e. 2 + 7 + 1 = 10, 4 + 3 + 3 = 10. Next, we add 10 + 10 = 20. Finally, we add 20 and 2 to get 22 as follows on the next page.
Method 5: Addition – Group Addends

Group recurring addends. Multiply each addend by the number of times it occurs. Add the products. Add any single addends. For example, \(4 + 2 + 4 + 5 + 6 + 5 = ?\) First we find 4 and 4 in one group, 5 and 5 in one group. Next, we multiply 4 by 2 and multiply 5 by 2, respectively. We add the products and add the rest of the single addends to get 26 as follows:

\[
\begin{align*}
\text{Problem} & & \text{Step 1} & & \text{Step 2} & & \text{Step 3} \\
4 & & 4 & & 4 \times 2 = 8 & & 8 + 10 = 18 & & 18 + 2 + 6 = 26 \\
2 & & 2 & & 2 \\
4 & & 4 \\
5 & & 5 \\
6 & & 6 \\
5 & & 5 \\
25 & & 25 \\
\end{align*}
\]

Fig. 5. Addition – Group Addends

Method 6: Addition – Group Addends with Like Sums

Find groups of addends who sums are the same number. Multiply this sum times the number of groups. Add any single addends. For example, \(3 + 4 + 2 + 1 + 3 + 4 + 2 = ?\) First we find 3 and 2 in one group, 1 and 4 in one group, 3 and 2 in one group. Add 3 and 2, 1 and 4, 3 and 2 respectively. We multiply 5 by 3. Add the products and the single addends to get 19 as follows:

\[
\begin{align*}
\text{Problem} & & \text{Step 1} & & \text{Step 2} & & \text{Step 3} & & \text{Step 4} \\
43 & & 43 & & 43 \\
23 & & 23 \\
67 & & 67 \\
31 & & 31 \\
52 & & 52 \\
87 & & 87 \\
+ 74 & & 74 \\
\end{align*}
\]

Fig. 6. Addition – Group Addends with Like Sums

Method 7: Addition – Add Compatible Numbers and Group by Place Value

For each place value, find digits whose sums are powers of 10. For each place value, multiply the power of 10 by the number of occurrences. For example, \(43 + 23 + 67 + 31 + 52 + 87 + 74 = ?\) In the ones column, we find 3 and 7 make ten and 1, 2 and 7 to make ten. That is two 10s, or 20. Next we add the rest of the single addends. In the tens column, we find 40, 60 in one group, 20 and 80 in one group, and 30 and 70 in one group. The sum of the numbers in each group is 100. Then we multiply 100 by 3 to get 300. Next, we add the rest of the single addends in this column. Finally, we add the sums in the ones column and the tens column to get an answer of 377 as follows:

Fig. 7. Addition – Add Compatible Numbers and Group by Place Value

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Method 8: Subtraction from Powers of Ten – Reference Number of 9

To do subtraction from powers of ten, first take 9 as a reference number. Step 1: Subtract each digit of the subtrahend from 9. (Subtrahend is a number that is to be deducted from another number.) For example, if the digit is 6, automatically think 3 because $9 - 6 = 3$. Step 2: Add 1 to the answer. For example, $1000 - 678 = ?$ First, take 9 as the reference number, you see 678 and you will think it as 321. Next, you just add 1. The answer of 322 is obtained as follows:

![Fig. 8. Subtraction from Powers of Ten – Reference Number of 9](image)

Method 9: Subtraction – Add a Power of 10

Subtract the subtrahend from the next larger power of 10. Add the resulting difference to the minuend. (Minuend is the number from which another number subtrahend is to be subtracted.) Subtract the power of 10. For example, $675 - 297 = ?$ First, we subtract 297 from 1000. We get 703. Find the sum of 675 add 703. Subtract 1000 from this sum to get 378 as follows:

![Fig. 9. Subtraction – Add a Power of 10](image)

Concluding Remarks

Computation is used not only in mathematics but in our daily life. Introducing alternate methods of quick calculation to students is an excellent way to help them develop number sense and mental computation abilities. It is also an excellent way for students to know how to work math numbers. Students will learn different ways of doing calculations. In addition, it is a creative and problem solving approach to number operations.

There are multiple algorithms for doing arithmetic, some come naturally and others are learned. Students should be acquainted with these multiple algorithms and should have competence in paper-and-pencil calculations. The algorithms can and should be taught and practiced. We encourage teachers to develop students’ speed and accuracy in computation. Careful planning and assessment allow children’s skills to develop and progress to develop number sense. In addition, students should be encouraged to develop and use their computational skills and learn to adopt strategies to help them solve everyday mathematical problems.

References


The Misfit of High School Math and Practical Math: Bridging the Divide

William M. Carroll – St. Ignatius College Prep, carroll_w@yahoo.com

During college, I recall stopping at a math professor’s office to find him occupied with a music student busily trying to design a cone that would serve as a mute for the musician’s trumpet. Over the years, I have occasionally wondered whether they actually found the cone’s specifications and constructed the mute. Two things struck me about the incident at the time: first, mathematics was actually being used to make something; second, the task didn’t appear to be trivial for the professor. Neither of these matched my general experience in school mathematics.

A few years later, I needed to rebuild the front stairs of my house. Being a math teacher, it seemed trigonometry would help me plan out the length of my stringer: All I needed were the angle of elevation and the height of the porch. Fortunately, I didn’t make my cuts from these calculations, but first cut out a cardboard model based on my math and tried fitting it to the porch. While I wasn’t completely off, it would have been a slightly lopsided stairway if I had based it on my math. The problem was not with my calculations. But I would argue it wasn’t with my measurement either, given the imprecise instruments I was using and my own lack of experience. Instead it illustrates a problem that is not often addressed in school: The alignment of school mathematics and practical mathematical application is not as smooth a fit as we teach.

Other examples illustrate how we can mislead students into thinking that mathematics can easily model the world: the bouncing ball to illustrate the sum of an infinite geometric sequence \((\frac{1}{2} + \frac{1}{4} + \ldots)\) and a free throw to illustrate a parabolic path. I’ve dropped many balls, but have never seen one bounce more than a dozen or so times, usually fewer. And while a parabola might approximate the path of a basketball, other forces like drag and spin degrade the path enough that it’s not parabolic. I try to tell my students that mathematics can model things, but that the model is only an approximation. Activities that help students realize that applied mathematics is not as precise and easy, as the curriculum often suggests, makes math more interesting and motivating.

I will describe two activities that I have used for several years with good success with my geometry students. They are given the opportunity to apply and practice the math they learned in class. The activities are heavy on measurement, planning, estimation, reasoning and collaboration, skills most students need to practice more. While each activity requires a class period, the time spent is valuable and helps students dig deeper into mathematics. It is hard to come away from these activities confident that you have “the correct answer,” but when well done, students should feel confident that they are approaching the correct answer. And that this is an aspect of mathematics.

Activity 1: Using Indirect Measurement to Compute Height or Distance

For this activity, students are taken to an area where there are trees or buildings
they can measure. They should know the basic trig relationships and know how to solve problems with them. A few tools are needed: protractor, tape measure, and calculators.

To prepare, a few student volunteers need to construct an angle measurement tool, one for each group of 3 – 5 students. You can use protractors, or copy the protractor in Figure 1 onto cardstock paper. The finished tool has a straw glued or taped to the protractor’s baseline, and a plumb line (a string) with a weight hanging freely from the center hole (Figure 2a). It is important to check this last point as students often attach the string in such a way that it does not hang freely but is blocked by tape, degrading the accuracy of the angle. Along with the angle finder, each group will need a linear measurement device like a 25-foot tape measure as well as calculators. My school is fortunate to have a clear view of the Sears Tower in Chicago and several tall buildings on campus, although a tree, lamppost, or tall building (Figure 2b) would work as well. Figure 3 shows the worksheet I pass out.

Usually no two groups get the same answer, and estimates of the distance and height vary quite a bit. I don’t worry about the students getting the correct answer (in fact, I’m not always sure what it is). Instead, the goal is that they made reasonable measurements of angles of elevation and distances, drew diagrams that included the necessary information, and applied correct mathematical relationships. A follow-up discussion in class is worthwhile, talking about problems of measurement, precision, and instrument error. It’s surprising how many students, even in an honors class, have difficulty using a protractor correctly or feeling comfortable with a tape measure. Some students are generally more adept at the mathematical symbolism, setting up equations or using trig relationships, than they are at the practical skills of measuring and estimating. But aren’t these practical mathematical skills at least as valuable for most people?

Final tip: A warm day in May is great for this activity.

Activity 2: Building 3-D Solids

For a number of years, I have given my students sets of 3-D solids, cones and spheres, pyramids and prisms, and asked them to find the total surface area and volume to the nearest tenth of an inch. This is a great way for them to apply the formulas they have learned: “Do I need the slant height or the altitude here?” While making decisions about measurement, “Is this edge 3 1/8 or 3 ¼ inches?” After our “heavier” class work with exact answers involving π and radicals, this gives students a chance to work with approximations or “hardware store numbers.” A set of large solids are needed for this activity.

More recently, I asked students to draw nets of 3-D solids and then construct the solids from these. The constraints on the project are all faces of the solid must be connected to another face, i.e., students can’t just cut out a rectangle and two circles of appropriate dimensions to make a cylinder. Faces should be drawn so that each touch one or more faces – that is, students should not cut out separate figures and tape them together. Planning how to attach circular faces to other faces can be especially challenging for the students. Students then cut out the figures and tape them together. Along with rulers and protractors, you will

1 As I tell the students, you would never go into a hardware store and ask for 3.2π feet of rope or a board √3 feet long.
need scissors and tape for each group. A heavier paper will make the figures sturdier.

I have found several aspects of this activity valuable. First, students must do some analysis and planning—what polygons make up a square pyramid. Second, the activity requires some planning, e.g., where should faces touch and connect. Sketches should be made ahead of time. Third, questions can be posed so that relationships between area and volume are related (Figure 4). Finally, measurement and group work are integral to the activity.

As a follow-up to constructing the solids, students can be asked to find the total surface area, volume, or other unstated measurement.

One of the nice aspects of this activity is that the finished products are not perfect. Prisms might be a little lopsided, pyramids don’t quite close up. Because this activity is their first attempt at such a construction, that is not important. It is like the math professor struggling with designing the correct cone for a trumpet. Moreover, I think it gives students the message that the figure they were attempting to make and the model they constructed are not exactly the same. Formal school mathematics approximates practical mathematics, but it takes time and practice for them to coincide. Their figures are not as perfect as the plastic 3-D figures we purchase from education suppliers, but students may learn more from this construction or acquire a different type of knowledge.

There are different ways of assigning the figures, depending on the level of your class. Figure 4 shows 3 different levels; generally I ask each group to construct two (one polyhedron and one circular solid) from a menu of choices I provide.

Summary

While word problems provide practice in forming equations, they can also delude the students into thinking the real world matches the mathematical world precisely. This is only true as people learn to make more precise measurements, use diagrams, and decide what information is important for completing a task, whether it is making curtains, designing a wooden compost bin, or constructing a conical mute. The activities above provide some ideas for what has worked for me, but many such activities can be designed to fit your curriculum. Compared to the standard train problems, lawn mowing problems, and rowing problems that are intended to make mathematics more real, these types of problems provide more practical, realistic examples of how people use mathematics. More importantly, activities like this can also help you assess students in areas like problem solving, considering the reasonableness of an answer, measuring and estimating, and other skills that classroom mathematics too often neglects—all areas that are recommended in the NCTM Principles and Standards.

Figures are on the next page.
Figure 1  Protractor for angle-distance measuring tool

Figure 2a. Constructing the angle measuring tool

Figure 2b  Using the angle measuring tool
Question 1: How far is the Sears Tower from the quad.
   The height of the Sears Tower, to the top of the antenna, is about 1720 ft. Use your Angle finder and trigonometry to estimate the distance to the Sears Tower. Make sure you include the following
   • A diagram showing the Sears Tower, you, and the height and angle labeled.
   • The math you use to get the answer.
   • Don’t forget to factor the eye-height of the person doing the measurement.
   • Round final answer to the nearest 10th of a mile.
   • You may want to make the angular measurement several times, until your group feels comfortable with it and agrees.

Question 2: How high is it to the peak of the roof of the gym?
   As in part 1 above, include a diagram and show the math you used in order to make this computation. In this case:
   ▪ Round to the nearest foot
   ▪ Don’t forget the distance from the ground to the protractor
   ▪ Make sure your group members agree on the measurements and the answer.

Figure 4
Directions for constructing right circular solids at different levels of difficulty

Easier: Measurements given
   1. Design and construct a cube with edges of 2.5 inches.
   2. Design and construct a cylinder with a diameter of 2 inches and a height of 3 inches.
   3. Design and construct a regular triangular pyramid with edges of 2 inches.

Challenging: Not all linear measurements given
   1. Design and construct a cube whose total surface area is 24 square inches.
   2. Design and construct a square pyramid with a base of 25 square inches and with lateral edges of 4 inches.
   3. Design and construct a cylinder with a base area of $4\pi$ square inches and a volume of $20\pi$ cubic inches.

More challenging:
   1. Design and construct a regular hexagonal pyramid with a base perimeter of 24 inches and lateral faces that are equilateral triangles.
   2. Construct a rectangular prism with a square base, a height of 3 inches, a volume of $19$ $\frac{1}{2}$ cubic inches.
   3. Construct a cone with a base circumference of about 7.5 inches and a lateral area of about 11.3 inches.
Fibonacci Pattern Formulas

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Teachers and their students often use the Fibonacci sequence when they study number patterns. The Fibonacci sequence is displayed below (\(a_n\) denotes the \(n^{th}\) term):

\[
\begin{align*}
    a_1 & = 1 \\
    a_2 & = 1 \\
    a_3 & = 2 \\
    a_4 & = 3 \\
    a_5 & = 5 \\
    a_6 & = 8 \\
    a_7 & = 13 \\
    a_8 & = 21 \\
    a_9 & = 34 \\
    a_{10} & = 55 \\
    \vdots & \vdots 
\end{align*}
\]

The formula used to generate this sequence is recursive. First, define \(a_1 = 1\), \(a_2 = 1\); from then on, \(a_n = a_{n-2} + a_{n-1}\). In other words, every term, after the 1st 2 terms, is equal to the sum of the two preceding terms (e.g. \(1+1 = 2\), \(1+2 = 3\), \(2+3 = 5\), \(\ldots\), \(21+34 = 55\), \(\ldots\)).

The Fibonacci sequence is a rich source for number patterns. Four patterns will be presented. In each case, students should be encouraged to find the pattern from the introductory numerical examples.

Pattern I.

Add the Fibonacci numbers as shown in the next column:

<table>
<thead>
<tr>
<th>Indicated Sum</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1+1</td>
<td>2</td>
</tr>
<tr>
<td>1+1+2</td>
<td>4</td>
</tr>
<tr>
<td>1+1+2+3</td>
<td>7</td>
</tr>
<tr>
<td>1+1+2+3+5</td>
<td>12</td>
</tr>
<tr>
<td>1+1+2+3+5+8</td>
<td>20</td>
</tr>
<tr>
<td>1+1+2+3+5+8+13</td>
<td>33</td>
</tr>
<tr>
<td>1+1+2+3+5+8+13+21</td>
<td>54</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

Note that each sum is one less than a Fibonacci number. In general, the sum \(a_1 + a_2 + a_3 + \cdots + a_n = a_{n+2} - 1\). For example, the sum of the first six Fibonacci numbers \((1+1+2+3+5+8)\) is 20; this is one less than the eighth Fibonacci number, which is 21.

Pattern II.

Square and add the Fibonacci numbers as shown:

<table>
<thead>
<tr>
<th>Indicated Sum</th>
<th>Sum</th>
<th>Indicated Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1^2</td>
<td>1</td>
<td>1 · 1</td>
</tr>
<tr>
<td>1^2 + 1^2</td>
<td>2</td>
<td>1 · 2</td>
</tr>
<tr>
<td>1^2 + 1^2 + 2^2</td>
<td>6</td>
<td>2 · 3</td>
</tr>
<tr>
<td>1^2 + 1^2 + 2^2 + 3^2</td>
<td>15</td>
<td>3 · 5</td>
</tr>
<tr>
<td>1^2 + 1^2 + 2^2 + 3^2 + 5^2</td>
<td>40</td>
<td>5 · 8</td>
</tr>
<tr>
<td>1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2</td>
<td>104</td>
<td>8 · 13</td>
</tr>
<tr>
<td>1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2</td>
<td>273</td>
<td>13 · 21</td>
</tr>
<tr>
<td>1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2</td>
<td>714</td>
<td>21 · 34</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td></td>
</tr>
</tbody>
</table>

Observe that each sum is the product of two consecutive Fibonacci numbers. In general, the sum \(a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2 = a_n \cdot a_{n+1}\). For example, the sum of the squares of the first six Fibonacci numbers

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(1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2) is 104; this is the product of the sixth and seventh numbers, which are 8 and 13.

Pattern III.

Multiply every other Fibonacci number as shown:

<table>
<thead>
<tr>
<th>Indicated Product</th>
<th>Product</th>
<th>Indicated Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1·2</td>
<td>2</td>
<td>1^2 + 1</td>
</tr>
<tr>
<td>1·3</td>
<td>3</td>
<td>2^2 - 1</td>
</tr>
<tr>
<td>2·5</td>
<td>10</td>
<td>3^2 + 1</td>
</tr>
<tr>
<td>3·8</td>
<td>24</td>
<td>5^2 - 1</td>
</tr>
<tr>
<td>5·13</td>
<td>65</td>
<td>8^2 + 1</td>
</tr>
<tr>
<td>8·21</td>
<td>168</td>
<td>13^2 - 1</td>
</tr>
<tr>
<td>13·34</td>
<td>442</td>
<td>21^2 + 1</td>
</tr>
<tr>
<td>21·55</td>
<td>1155</td>
<td>34^2 - 1</td>
</tr>
</tbody>
</table>

Note that the products are alternately one greater and one less than the square of a Fibonacci number. In general, the product

\[ a_n \cdot a_{n+2} \cdot a_{n+4} = \begin{cases} (a_{n+1})^2 + 1, & \text{if } n \text{ is odd} \\ (a_{n+1})^2 - 1, & \text{if } n \text{ is even} \end{cases} \]

For example, the product of the fifth and seventh Fibonacci numbers (5 and 13) is 65; this is one more than the square of the sixth Fibonacci number, which is 8^2 = 64. Also the product of the sixth and eighth Fibonacci numbers (8 and 21) is 168; this is one less than the square of the seventh Fibonacci number, which is 13^2 = 169.

Pattern IV.

Multiply triples of every other Fibonacci number, as shown in the next column:

<table>
<thead>
<tr>
<th>Indicated Product</th>
<th>Product</th>
<th>Indicated Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1·2</td>
<td>10</td>
<td>2^3 + 2</td>
</tr>
<tr>
<td>1·3</td>
<td>24</td>
<td>3^3 - 3</td>
</tr>
<tr>
<td>2·5</td>
<td>130</td>
<td>5^3 + 5</td>
</tr>
<tr>
<td>3·8</td>
<td>504</td>
<td>8^3 - 8</td>
</tr>
<tr>
<td>5·13</td>
<td>2210</td>
<td>13^3 + 13</td>
</tr>
<tr>
<td>8·21</td>
<td>9240</td>
<td>21^3 - 21</td>
</tr>
<tr>
<td>13·34</td>
<td>39,338</td>
<td>34^3 + 34</td>
</tr>
<tr>
<td>21·55</td>
<td>166,320</td>
<td>55^3 - 55</td>
</tr>
</tbody>
</table>

Note that the products differ from the cube of a Fibonacci number by that same Fibonacci number. In general, the product

\[ a_n \cdot a_{n+2} \cdot a_{n+4} = \begin{cases} (a_{n+1})^3 + a_{n+2}, & \text{if } n \text{ is odd} \\ (a_{n+1})^3 - a_{n+2}, & \text{if } n \text{ is even} \end{cases} \]

For example, the product of the fifth, seventh and ninth Fibonacci numbers (5, 13 and 34) is 2,210; this is 13 more than the cube of the seventh Fibonacci number, which is 13^3 = 2,197. Also, the product of the sixth, eighth and tenth Fibonacci numbers (8, 21 and 55) is 9,240; this is 21 less than the cube of the eighth Fibonacci number, which is 21^3 = 9,261.

Possible Challenges/Extensions:

Challenge 1
Verify these patterns using mathematical arguments, i.e. math induction and geometrical representations.

Outline of Solution
For instance, let us use mathematical induction to establish the formula resulting from Pattern I. We wish to prove that

\[ a_1 + a_2 + a_3 + \ldots + a_n = a_{n+2} - 1 \]

If \( n = 1 \), the formula would state that \( a_1 = a_3 - 1 \). Since \( 1 = 2 - 1 \), this is true. Assume next that the formula is true for \( n = k \); this would state
that \( a_1 + a_2 + a_3 + \cdots + a_k = a_{k+2} - 1 \). We must now establish that the formula is true for \( n = k + 1 \). We consider the sum

\[
(a_i + a_2 + a_3 + \cdots + a_k) + a_{k+1}
\]

\( = a_{k+1} + a_{k+2} - 1 \) (by the inductive hypothesis)

\( = a_{k+3} - 1 \) (by the Fibonacci definition)

The formula is established.

Challenge 2

Experiment with the Fibonacci and other sequences to find other number patterns. There are literally hundreds of these patterns lying “right beneath the surface.”

For instance:

A. Let \( a_n \) be the \( n^{th} \) Fibonacci number. Several such explorations are shown.

1. \( a_1 + a_3 + a_6 + a_7 + \cdots + a_{2n-1} = ? \)
2. \( a_2 + a_4 + a_8 + \cdots + a_{2n} = ? \)
3. \( a_1 + a_4 + a_7 + a_{10} + \cdots + a_{5n-2} = ? \)

B. Let \( L_1 = 1, \ L_2 = 3, \ L_n = L_{n-1} + L_{n-2} \) for \( n \geq 3 \). These are special Fibonacci Type numbers called Lucas numbers. Find similar relationships for Lucas numbers for Patterns 1-4.

C. Explore geometric patterns. This is a representation of pattern II.

\[
\begin{array}{c}
1 \\
\hline
1
\end{array}
\]

area = 1

\( 1 = a_1 \cdot a_2 \)

\[
\begin{array}{c}
1 \\
\hline
1
\end{array}
\]

area = 2

\( 2 = a_2 \cdot a_3 \)

\[
\begin{array}{c}
2 \\
\hline
1
\end{array}
\]

area = 6

\( 6 = a_3 \cdot a_4 \)

\[
\begin{array}{c}
3 \\
\hline
2
\end{array}
\]

area = 15

\( 15 = a_4 \cdot a_5 \)
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