THE ILLINOIS MATHEMATICS TEACHER

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From the Editors…

Welcome to the Spring issue of the Illinois Mathematics Teacher. We wish you the best of luck as the school year winds down to a close.

This issue of the IMT once again contains many useful and interesting articles and activities covering a wide range of topics. “Walt Disney’s Math-O-Mart” is a great article about a store that was set up and how it is used at the elementary and middle school level. James Rauff’s article gives a new perspective to helping students understand how to use proofs. William Beggs and Dane Camp collaborated to share their process in finding the solution to a question involving dice. “A Fun Polygon Activity” and “Making Content Relevant to Students: iPod Playlists and Permutations” both include activities you might want to incorporate into your classes next year. “Watson and the Mysterious Number One” relates an interesting geometry problem. “Milk Chocolate M&M Color Distribution: A Chi-Square Experience” provides a tasty new activity to try with candy. If you are interested in lesson study, Mary T. McMahon’s article will definitely be of interest to you. Finally, Denise Reid and Sudhir Goel have included a great article dealing with equations of lines.

Included at the end of this issue is a form for becoming a reviewer for the IMT. In order to ensure that the articles are of interest to our readers, we send them to reviewers to get their approval. You may notice that some reviewers’ names have been deleted, mostly due to a lack of an email address or to email bounce backs indicating an incorrect email address. We need your help to rebuild our list of possible reviewers. We would love to have reviewers willing to review in only one area or multiple areas. As postage prices continue to rise, we are trying to conduct most of our communications by email, so please update your information and your email address as well.

We would appreciate hearing from any of you out there who are reading the journal. We would especially like to hear about any activities from the IMT that you have used with your students. Your comments and constructive criticism are heartily solicited.

Please consider submitting an article or classroom activity to the IMT. Your articles are needed to continue the sharing of ideas and the publishing of this journal.

Thank you for sharing.

Marilyn and Tammy editors
Walt Disney’s Math-O-Mart
Ann Wallace, College of Charleston, wallacea@cofc.edu
Annie Shear (pictured), Walt Disney Magnet School, annies@comcast.net

Walt Disney School in Chicago, Illinois is a magnet school whose philosophy is multi-cultural education. It has an open classroom approach and houses grades Pre-Kindergarten through eight. There are eight classes per grade and a total of eighteen hundred students in the school. There is a multicultural lottery system to attend, and as many as twenty different cultures and different languages are found at the school. Teaching jobs are very competitive. Applicants must go through rigorous interviews, classroom observations, and oral exams to be appointed by a board.

The first time I walked into the Communication-Art-Center I immediately felt like I was walking into an actual grocery store (Figure 1). There were aisles stacked with grocery products, freezer cases full of goods, meat and produce departments, checkout lanes with cash registers, a bakery, a pharmacy and a customer service center. I was amazed to see the second-grade employees running around busily stocking and restocking shelves, totaling groceries, taking money, bagging groceries, and assisting customers.

I soon learned of the long journey taken to get here. I was visiting Disney’s Math-O-Mart in its sixth year, and it had been an evolving process. The mastermind behind Math-O-Mart is a second-grade teacher named Annie Shear. As I talked with Ms. Shear I found she was increasingly frustrated with her students’ inability to ‘add up’ to make change. She explained, “If ice cream bars cost $2.93 and you pay for them with a $5 bill, my students couldn’t go two ninety-four, two ninety-five, and a nickel makes $3.00, another dollar is $4.00 and one more dollar makes $5.00.” To help teach this concept she gave each of her students a set of play money and displayed a picture of an item (such as ice cream bars) that cost less than $5.00. She asked her students to use their money to show her how much change they would receive if they paid for the ice cream bars with a $5.00 bill. This led her to the idea of creating a shopping experience for her students where they would be the ones responsible for making change from purchases others made. She reasoned that if her students were the ones making change, they would learn the concept. She thought of the grocery store concept because shopping is familiar to children as well as an excellent example of a place where mathematics is put to work in the ‘real world.’

With this larger idea in mind, she wrote and was awarded a grant from the Chicago Foundation for Education in the amount of $400. She used the money to buy one cash register, register tape, play money, posters and markers to set up a ‘pretend’ grocery store. Ms. Shear organized the help of parents to contribute household items such as empty cereal boxes, frozen food boxes, laundry detergent, pet products, etc. (no glass products) to stock the grocery store. She borrowed plastic fruit and fake bread from the pre-school. To get other items, such as grocery baskets, smocks, shelves and checkout lanes she “schmoozed” local stores who subsequently donated them. She also used her own money to help finance the project.
Ms. Shear explained to the students and emphasized the importance of their participation. She posted the needed positions, and students were required to fill out an employment application (Figure 2). The positions included cashiers, managers, stock runners, customer service help, pharmacist, price checkers, callers, and labelers (Figure 3). She added, “At first they all wanted to be cashiers…until they realized how hard it was going to be. Just punching in the prices is time consuming, and if the customers were talking to them while they were trying to make change, they might get rattled.” They really had to understand what all the positions entailed before deciding which one was best suited for them. For example, the greeter had to make all the customers feel comfortable as they walked through the door, and the pharmacist needed to help customers locate appropriate vitamins, read labels and find first-aid. Cashiers had to key in the groceries, make appropriate change, and keep up with the discounts and specials that changed each day. The stock people had to know where everything goes. The stock runners had to hurry to put all the food and other products back on the shelves between one class leaving and the next class arriving (Figure 4). I had an opportunity to interview the ‘head stock’ whose job was to make sure the shelves were properly stocked. When I asked him about his job he replied, “[I’m] head stock…I have my own aisle…I have to look after people like if they’re doing something wrong and stuff…if they aren’t stocking right or if it gets backed up.”

Ms. Shear stated, “I had to suggest to some students what a good choice of job would be.” On the job application the students had to include their qualifications for different jobs so they really had to think about what the job entailed and what they had to offer. For example, one student who wanted to be a cashier wrote, “I am responsible, reliable and good at math.” Ms. Shear reasoned, “If they couldn’t answer [qualifications], why would they be hired?” The students were expected to fill out the application as if they were filling out an actual job application – if they used poor grammar or misspelled words they were not hired.

Once the positions were filled, students had to be trained to perform their jobs, which Ms. Shear said was very difficult for the students to learn. She later relied on students who had previously held the jobs to help with the training. She explained, “Once the store managers and assistant managers are trained, they are in charge. If there is a problem, that’s who they go to, not me.”

After the first grocery store was created and the students were trained to perform jobs, they had to have customers. Ms. Shear invited another second-grade class to shop at the Math-O-Mart. As other teachers began to see what was going on, they wanted to bring their classes to the Math-O-Mart too. More and more students wanted to participate, and eventually the whole school became involved, so it was moved to the large, open Communication Art Center.

After the entire school became involved, a local grocery store (Jewel-Osco) assisted by setting up a model of a real grocery store including checkout lanes, actual store shelves and freezers, shopping carts, scales, and smocks for the student workers. Ms. Shear enlisted the help of second-grade team member Cheryl Henry, who helped with scheduling of all the classes, and her students participated with the store operation.

Once the Math-O-Mart was ready for customers, all of the classes from Pre-Kindergarten to grade 8 were invited to shop. They could pay for their purchases with play money and receive change and a
receipt verifying their purchases. The Math-O-Mart was meant as a learning tool, and lesson plans for each class or entire grade level were encouraged. “The teachers bring their classes because they are able to teach a variety of curricular activities, standards, and skills at once,” said Ms. Shear.

Six years after Annie Shear created the Math-O-Mart, I was invited to its opening day. I was welcomed by third-grade student Carey who, for the second year, delivered the greeter speech. Hi. Welcome to Disney Math-O-Mart. You will each receive some money to shop with. After you receive your money to shop with, you will get a cart or a basket. After you get your cart or basket, you will be able to shop in the store. Please...no skipping, running or jumping in the store. Shop as you wish, but don’t go overboard! After you make all your selections, get in line to pay. If you receive any change, please drop it on the tray at the end of the aisle. Wait for your teacher on the carpet. Thank you for shopping at the Walt Disney Math-O-Mart. I hope you will enjoy your shopping experience.

Teachers took advantage of this incredible opportunity and wrote lesson plans to support a variety of skills and content standards. Letters were sent to parents prior to children shopping at the Math-O-Mart asking them to familiarize their children with a grocery store. In the following table we have included a list of some of the activities and assignments from each grade level that teachers planned:

<table>
<thead>
<tr>
<th>Grade</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-Kindergarten</td>
<td>Pre-K teachers visited the Math-O-Mart prior to bringing their students and determined some ‘price adjustments’ to help their students. Four-year-old students shopped in pairs (Figure 5). Each pair was given a play $1 bill. Their task was to purchase an item(s) without spending more than $1 (if possible). After making their purchases the students were asked to identify the coins they received as change. These students did not understand why they had to leave the items after they made their purchases. They wanted to take everything home with them!</td>
</tr>
<tr>
<td>Kindergarten</td>
<td>A Kindergarten teacher visited the Math-O-Mart before its opening and determined items she believed her students would recognize and then created a ‘Shopping List’ (Figure 6) with pictures of various products. Students were to shop for the items and to circle them if they found them or put an ‘x’ on an item they could not find. This gave them practice using a shopping list. After they returned to the classroom, the students had to tell how many items they found, did not find and how much money they had spent.</td>
</tr>
<tr>
<td>First Grade</td>
<td>This grade was introduced to four 3-dimensional shapes (cone, cylinder, cube and rectangular prism) before they visited the Math-O-Mart. Students in this class were asked to purchase at least one item of each shape. They compared similar and different items once they were back in their classroom. For example, a soup can and an oatmeal container both represented cylinders (Figure 7). They were also asked how they could sort their items by different characteristics such as ‘items that roll.’</td>
</tr>
<tr>
<td>Second Grade</td>
<td>One second-grade teacher asked her students to help with her shopping list that she created</td>
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</tbody>
</table>
from a store circular. She told her students that she would like to know ‘about’ how much her groceries would cost. She provided a store circular from a local grocery store to aid in their estimations. Next, she asked her students to create their own shopping list of five items and to estimate its cost. The students purchased their items and then compared the actual cost to their estimation. Another second-grade teacher asked his students to make a shopping list if they had to purchase items for their own birthday party. What would they need? This gave students the opportunity and experience to really plan. The teacher prompted their lists by making comments and asking questions such as, “I’m glad you’re planning to have a cake. What do you need to serve it? What do you need to eat it? If you have ice cream, do you need a utensil other than a fork?”

<table>
<thead>
<tr>
<th>Grade</th>
<th>Activity</th>
</tr>
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<tbody>
<tr>
<td>Third Grade</td>
<td>A third-grade teacher discovered that many of her students did not know the difference between a name brand and a generic brand (Figure 8). Her activity had the students finding the difference of the cost between name brand and generic brand of items as well as comparing different name brand items. Afterwards, she had a ‘taste test’ to compare several of the items (e.g. Coke®, Pepsi®, and generic cola; Heinz®, Hunts®, and generic ketchup; Kraft®, Velveeta®, and generic macaroni), tallied the votes for the preferred products and then created a class graph for Preferred Products. In addition,</td>
</tr>
<tr>
<td>Fourth Grade</td>
<td>A class was asked to make a balanced food plan for one day using the Food Pyramid (Figure 9). They were only allowed to purchase 20 items, and the items had to represent the correct fractional part of the food pyramid. For example, 1/5 (4/20) of the Food Pyramid is vegetables, so the students would purchase 4 vegetables. Another class integrated the grocery store concept with Social Studies when studying other countries. Groups of students were assigned a particular country and part of their research included identifying different staple foods common to that country. The students had to conduct research on their selected country to find out what foods the people ate. They had to plan a meal, make a shopping list and subsequently shop for the foods at the Math-O-Mart. Prior to assigning the different countries, the teacher visited the store to make sure certain goods were available.</td>
</tr>
</tbody>
</table>
servings, how many calories, fat, etc. per serving so that they would recognize how serving size affects the contents of the product. During their shopping excursion students answered questions such as, “Which brand of cereal is healthy and affordable?”

### Sixth Grade

One class learned how to adjust the cost of a shopping list based on store sales and coupons as well as to calculate the appropriate sales tax (8.5%). Another class made the ‘dinner to go’ items available for purchase (Figure 10). Instead of just thinking about what foods they personally liked, they had to consider what made a balanced meal. To do this they examined the six food groups to help determine what would make up a healthy meal.

### Seventh Grade

Students had to ‘make’ dinner for four people. They first had to decide on the menu, and then make their shopping list before going to the Math-O-Mart. I asked an excited student what dinner he planned to make and he replied, “Tacos! With lettuce, tomato, and cheese…like we have corn with it…and cookies and milk for dessert.” I asked him if it was a meal he had at home and he said, “U-huh. It cost a lot!” Other students had to estimate how much they would spend on a meal they were preparing for four. Jeremy, a student customer said, “It was embarrassing when you’d hear ‘Over ring on lane 3’ and know they were talking about you! We had to figure out what to put back. We weren’t thinking.”

### Eighth Grade

Some eighth-grade students made checks (Figure 11) for seventh- and eight-grade students to write and to pay for their purchases. Some also interviewed the student employees. They asked questions such as ‘What do you like about being a manager?’ They incorporated the interviews into a Language Arts assignment and wrote journal articles. They made up their own books with construction paper and created the Disney Star and the Disney Astrologer, which were available for customers to read in the check out line and in the pharmacy waiting area.

Ms. Shear reflects on the progression of Math-O-Mart, “Students who have been here for 5 years know the routine. One student in second grade didn’t want to work, but now, he’s in fifth grade, and I can’t run the store without him! To see him grow as an individual and a responsible person…it’s phenomenal and the children are so proud of themselves.” One parent added, “It’s unbelievably fun. It teaches [children] how to shop. It teaches them the value of a dollar and you know what Mom and Dad are doing at the store.”

Ms. Shear continued, “The kids love it. Everyone looks forward to it. Everyone wants to work at it. Everyone is so tired at the end of the day. It’s a real job experience. From the first day of school people are asking when Math-O-Mart will be. Students who did not appear to be hard workers surprisingly exhibit such enthusiasm in this project and take this job seriously. You learn a lot about the students. The children are respected in their jobs by their peers. There is no teasing.”
Conclusion

Teachers are continually urged to integrate their curriculum in a manner that is both developmentally sound and motivating. Money is a common focus of people everywhere. It is of natural interest to children from an early age. Whether children work for money and/or receive it in the form of allowances, they learn the function of money in obtaining things that are needed or wanted. This integrated curriculum project demonstrates the more practical aspects of mathematics as well as developing advanced planning skills and responsible work habits. An experience, such as Disney’s Math-O-Mart, that engages children in related social studies, science, mathematics, and language arts activities, gives students real-life experiences in everything from shopping on a budget to employment skills and management responsibilities.
Figure 9

Figure 10

Figure 11
How can students get experience in formal proof when their mathematical abilities or experiences are limited? One technique that I’ve been able to use successfully involves viewing human languages (or non-human languages like Klingon or Sindarin) as axiomatic systems.

Axiomatic systems may be described as consisting of four major components: definitions, axioms, theorems and an underlying logic that provides rules for deriving theorems from axioms and other theorems. When we use language systems as examples of axiomatic systems, we have definitions and an underlying logic that includes system-specific rules of inference, i.e. the rules for moving from definitions to theorems. If we think of languages as axiomatic systems we can view sentences as theorems in the system resulting from the correct application of the rules of inference.

Students need to understand that su is a nom because it is defined to be a nom and that any assertion about su must ultimately refer back to this definition. For many students this surface arbitrariness of definition is an important component of proof to which little attention is paid.

Next we will specify a rule of inference. Essentially, a rule of inference allows us to write new strings of symbols from given strings. For our language subsets we use constructive rules of inference. They will tell us what we are permitted to do with strings. Here is a rule of inference for Swahili.

SR1. One may attach the string ki or the string vi to the left of any nom. The result is a theorem.

Swahili is a language spoken in East Africa by more than 45 million people. I use a small subset of Swahili (Perrott, 1978) to introduce the axiomatic method to my students.

We begin with a definition. The definitions we make are simple categorizations of strings of letters. These Swahili strings obviously have meaning, but we don’t need to know what the strings of letters mean in order to prove theorems.

Definition SD1. A nom is any of the strings su, tabu, atu, kapu, and jiko.
(If we are comfortable with set notation then we could say instead that a nom is any element of the set {su, tabu, atu, kapu, jiko}.)

Students need to understand that su is a nom because it is defined to be a nom and that any assertion about su must ultimately refer back to this definition. For many students this surface arbitrariness of definition is an important component of proof to which little attention is paid.

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SR1. One may attach the string ki or the string vi to the left of any nom. The result is a theorem.
This rule of inference is simple and unambiguous. As a result, it is easily used in proving theorems. Theorems result from definitions, axioms and other theorems by means of the rules of inference. In our language fragments, the theorems will be new strings of symbols. We can now state and prove the following theorem. The justification for each step in the proof is given parenthetically.

Swahili Theorem 1 (ST1). \textit{kisu}.
Proof.
1. \textit{su} is a \textit{nom} (Definition SD1)
2. \textit{kisu} (Rule of inference SR1 from #1)

We can also prove this theorem.

ST2. \textit{visu}.
Proof.
1. \textit{su} is a \textit{nom} (Definition SD1)
2. \textit{visu} (Rule of inference SR1 from #1)

Students have no trouble stating and proving the Swahili theorems \textit{vitabu} and \textit{kijiko} and many others. They may also be able to calculate how many theorems can result from these rules!

These theorems and proofs are strictly formal. The student does not have to deal with the meaning of the Swahili words at all. Using a language fragment devoid of semantics forces the student to focus on the structure of the argument.

Now let’s add another definition and modify our rule of inference to handle a new set of symbols.

Definition SD2. An \textit{ajem} is any of the strings \textit{kubwa, dogo, refu, zuri, and baya}.

SR1 (version 2). One may attach the string \textit{ki} or the string \textit{vi} to the left of any \textit{nom} or to the left of any \textit{ajem}. The result is a theorem.

We can now prove the theorem.

ST5. \textit{kikubwa}
Proof.
1. \textit{kubwa} is an \textit{ajem} (Definition SD2)
2. \textit{kikubwa} (Rule of inference SR1 from #1)

Theorem proving is an important mathematical skill, but so is theorem-conjecture. With the Swahili axiomatic system we have so far, we can ask students to formulate and prove new theorems. Almost every student will easily state and prove a theorem like \textit{vidogu}.

They will also be able to tell that \textit{vivi} and \textit{kubwaki} are not theorems and explain why they are not. Specifically, \textit{vivi} is not a theorem because no part of it is either a \textit{nom} or an \textit{ajem}. \textit{kubwaki} is not a theorem because the \textit{ki} is attached to the right of \textit{kubwa} instead of the left as required by SR1. These simple exercises reinforce the idea that a theorem must be deduced from definitions or axioms using rules of inference.

Our rule of inference may also be expressed as an if-then statement like this.

SR1 (version 3). If \(x\) is a \textit{nom} or \(x\) is an \textit{ajem} then \(\text{ki}x\) and \(\text{vi}x\) are theorems.

Now let’s add some complexity to our system with the addition of a new rule of inference. The symbol \(\bigtriangleup\) indicates an obligatory space.

SR2. If \(x\) is a \textit{nom} and \(y\) is an \textit{ajem} then both \(\text{vi}x\bigtriangleup\text{vi}y\) and \(\text{ki}x\bigtriangleup\text{ki}y\) are theorems.

We can now state and prove the theorems.

ST6. \textit{vitabu vikubwa}.
Proof.
1. \textit{tabu} is a \textit{nom}. (SD1)
2. \textit{kubwa} is an \textit{ajem}. (SD2)
3. \textit{vitabu vikubwa} (SR2 from #1 and #2)
ST7. **kikapu kizuri**

Proof.

1. **kapu** is a *nom*. (SD1)
2. **zuri** is an *ajem*. (SD2)
3. **kikapu kizuri**. (SR2 from #1 and #2).

Students will be able to conjecture and prove many other theorems using these rules and definitions.

Although, we have been stressing the formal aspect of language, we mustn’t disregard the obvious fact that these Swahili theorems mean something. Students will be keen to know what the Swahili words mean in English. The theorems we have been proving are simple Swahili noun phrases. Number is indicated by the prefixes *ki-* (singular) and *vi-* (plural). In Swahili the adjective follows the noun it modifies.

Here are the adjectives with English translation: **kubwa** (*big*), **dogo** (*small*), **refu** (*long*), **zuri** (*good*), **baya** (*bad*). Here are the nouns: **su** (*knife*), **tabu** (*book*), **atu** (*shoe*), **kapu** (*basket*), **jiko** (*spoon*).

Thus, **vitabu vikubwa** may be translated into English as “the big books” and **kikapu kizuri** may be translated as “the good basket”.

We can make good use of the English translations of foreign language phrases to give students practice in another important aspect of mathematical theorem proving – conjecture. I’ll illustrate how this can be done with a language from Central America, Quiche.

Quiche is a language spoken by about 650,000 people in Guatemala (Mondloch, 1978). It is one of the languages of the ancient Maya. Students can develop the analytical skills needed to formulate theories by analyzing words, phrases, and sentences in an unfamiliar language. To help students acquire these skills, I present a series of questions that will lead them to formulating their own axiomatic system.

<table>
<thead>
<tr>
<th>Quiche</th>
<th>English</th>
<th>Quiche</th>
<th>English</th>
</tr>
</thead>
<tbody>
<tr>
<td>ja</td>
<td>house</td>
<td>balam</td>
<td>jaguar</td>
</tr>
<tr>
<td>tz'iquin</td>
<td>bird</td>
<td>caxlan</td>
<td>foreign</td>
</tr>
<tr>
<td>nim</td>
<td>big</td>
<td>cumatz</td>
<td>snake</td>
</tr>
<tr>
<td>co</td>
<td>strong</td>
<td>chak'</td>
<td>cooked</td>
</tr>
<tr>
<td>ch'uti'n</td>
<td>small</td>
<td>joron</td>
<td>cold</td>
</tr>
<tr>
<td>tinamit</td>
<td>town</td>
<td>lej</td>
<td>tortilla</td>
</tr>
<tr>
<td>juyub</td>
<td>mountain</td>
<td>lol</td>
<td>grasshopper</td>
</tr>
<tr>
<td>chicop</td>
<td>animal</td>
<td>patzapic</td>
<td>shaggy</td>
</tr>
<tr>
<td>utiw</td>
<td>coyote</td>
<td>abaj</td>
<td>rock</td>
</tr>
<tr>
<td>quej</td>
<td>horse</td>
<td>bak</td>
<td>thin</td>
</tr>
</tbody>
</table>
Working in groups, the students must decide whether or not each of the following is a valid Quiche sentence. They must also explain what is wrong with those that are not valid.

a. ch'uti'in lē tinamit (valid)
b. quej lē bak (not valid because the noun precedes the adjective)
c. lē juyub sak (not valid because lē shouldn’t be first)
d. bak lē sak (not valid because there is no noun)
e. patzapic lē lol (valid)
f. balam lē quej (not valid because there is no adjective)

Notice that this process requires students to articulate the Quiche grammar rule they have discovered. The next step is to write the rule. Most students will come up with something like this:

Rule of Inference QR1. A Quiche theorem may be constructed by writing an adjective followed by lē followed by a noun.

Most will not define “noun” or “adjective”. So after some discussion and prodding they will produce the set-theoretic definitions.

Definition QD1. Each element of the set {tz’i, che’, ja, tinamit, tz’iquin, juyub, chicop, utiw, quej, abaj, lol} is a noun.

Definition QD2. Each element of the set {ke’k, sak, cak, rax, nim, co, ch’uti’n, bak, joron} is an adjective.

With the formal definitions and rule of inference in hand, students can now prove the following theorems.

QT1. k’ek lē tz’i
QT2. sak lē tz’i
QT3. nim lē juyub

Students find the previous examples relatively easy to handle. However, one of the wonderful features of human language is its complexity. So, more advanced problems can be posed. For example, consider these sentences in Comanche, a language of North America (Godby, Wallace and Jolly, 1982).

paruu ku hunuru nohiniyu
The raccoon was playing around in the creek.

wasape hunuru nohiniyu
The bear was playing around in the creek.

paruu ku paaru nohiniyu
The raccoon was dancing around in the creek.

paruu ku hunu hunukuhpaiki nohiniyu
The raccoon was playing around through the creek.

paruu ku hunuru nihkanliyu
The raccoon was playing in the creek.

wasape hunukuhpaiki sariia miakiiyu
The bear was chasing the dog through the creek.

sarii kwasinavoo kihtsiaayu
The dog was biting the snake.

It’ll take some work, but eventually students working together will be able to devise definitions and rules that allow them to prove this Comanche theorem and translate it into English.

CT1. sarii hunukuhpaiki paaru ku miakiiyu
The patterns of human languages can serve as useful examples of theorem formation and proof for students. They can reach the point of conjecture and proof with natural languages in a short period of time. Often those students who struggle with algorithmic mathematics can find success in proving language theorems. It is then an easier transition to mathematical proof.

Analyzing language structures has the added feature of providing connections between mathematics and a wide range of disciplines in the humanities and social sciences. Keep in mind that teaching a foreign language is not the goal. Rather, we use the formal structure of a language as an example of an axiomatic system.

References


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Prospective authors should send:

- **Five (5) copies of your article**, typed, double-spaced, 1-inch margins. Put your name and address only in the cover letter. No identifying information should be contained in copies of the manuscript. Articles should be no more than ten pages in length, including any graphics or supplementary materials.

- **A diskette with your article, including any graphics**. We prefer that the article be written in Microsoft Word and that it be saved on an IBM-compatible disk. Graphics should be computer-generated or drawn in black ink and fit on an 8 ½"×11" page.

- **Your name, address, phone, and e-mail** (if available) should be included in a cover letter.

- **A photo of yourself (Illinois authors only)**, color or black/white.

To: Marilyn Hasty and Tammy Voepel
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What if Not? : Connecting Probabilities Involving the Sum of Dice

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Dane Camp, New Trier High School, campd@newtrier.k12.il.us

In the study of probability at the high school level, most students will solve problems involving the sum of two dice. A typical problem might be to find the probability that the sum is eleven when two dice are tossed. Teachers will often vary this problem by asking students to find the probability that the sum is less than eleven, or that the sum is seven or eleven. A creative way to generate more challenging problems is to use the “What if Not?” technique.

Stephen Brown and Marion Walter, at Harvard University, in the late nineteen-sixties developed this technique. The “What if Not?” technique begins by listing attributes of a given problem, theorem, or puzzle. For example consider the opening problem:

What is the probability when rolling two six-faced dice that the sum is eleven?

Six attributes for this problem are listed below.

1. Two dice are being tossed.
2. The sum is eleven.
3. The dice have six faces.
4. The sum is being calculated.
5. The dice are fair.
6. The number system is base ten.

The second stage of this technique is to negate these attributes one at a time and then, based on each negation, attempt to generate a new problem. For example suppose that the first attribute is negated. New problems would deal with sums of eleven where more than two dice are being tossed. A possible way to pose a new problem is as follows:

In a math classroom there are eleven six-sided dice sitting on top of a desk. A student enters the classroom and selects some of these dice from the desk. The student rolls the dice and obtains a sum. If the student wants the greatest probability of rolling a sum of eleven, then how many of these dice should the student select?

I have chosen to include the case of tossing two dice with the other possible cases. It is impossible with a single die to attain a value of eleven, and with more than eleven dice the sum will always be larger than eleven. The remainder of this article will tell the story that unfolded in an attempt to find a solution to this problem.

When I originally solved this problem I started by looking at ordered n-tuples and used permutations to help calculate these probabilities. For example with the case of three dice, I first found six triples where the sum was eleven. The six triples were \((1, 4, 6)\), \((1, 5, 5)\), \((2, 3, 6)\), \((2, 4, 5)\), \((3, 3, 5)\), and \((3, 4, 4)\). I then found all ordered triples that could be generated from these original six. The chart on the next page shows the 27 possible solutions for which the sum of three dice is eleven.
(Table 1)

Original Triple

New Triples

Counting Principle

Number of Dice

Probability

3-

Significant

Digits

(1, 4, 6)

(1, 6, 4), (4, 1, 6),
(4, 6, 1), (6, 1, 4),
(6, 4, 1)

3! = 6

2

\[ \frac{2}{36} = \frac{1}{18} \]

0.0556

(1, 5, 5)

(5, 1, 5), (5, 5, 1)

3! = 6

\[ \frac{3!}{2!} = 3 \]

3

\[ \frac{27}{216} = \frac{1}{8} \]

0.125

(2, 3, 6)

(2, 6, 3), (3, 2, 6),
(3, 6, 2), (6, 2, 3),
(6, 3, 2)

3! = 6

4

\[ \frac{104}{1296} = \frac{13}{162} \]

0.0802

(2, 4, 5)

(2, 5, 4), (4, 2, 5),
(4, 5, 2), (5, 2, 4),
(5, 4, 2)

3! = 6

5

\[ \frac{205}{7776} \]

0.0264

(3, 3, 5)

(3, 5, 3), (5, 3, 3)

3! = 3

\[ \frac{3!}{2!} = \frac{3}{2} \]

6

\[ \frac{4}{36} = \frac{1}{9} \]

0.00540

(3, 4, 4)

(4, 3, 4), (4, 4, 3)

3! = 3

\[ \frac{3!}{2!} = \frac{3}{2} \]

7

\[ \frac{210}{6^7} \]

7.50 \times 10^{-4}

(4, 3, 3), (3, 3, 4)

8

\[ \frac{120}{6^8} \]

7.14 \times 10^{-5}

(4, 4, 3), (3, 4, 4)

9

\[ \frac{45}{6^9} \]

4.47 \times 10^{-6}

(5, 2, 3), (2, 5, 3),
(2, 3, 5), (3, 5, 2),
(3, 2, 5), (5, 3, 2)

10

\[ \frac{10}{6^{10}} \]

1.65 \times 10^{-7}

(5, 3, 3), (3, 5, 3),
(3, 3, 5), (5, 3, 3),
(3, 3, 5), (5, 3, 3)

11

\[ \frac{1}{6^{11}} \]

2.76 \times 10^{-9}

The number of ways that the three dice can fall is \( 6^3 \) or 216. The probability of tossing three dice and having a sum of 11 is \( \frac{27}{216} \) or \( \frac{1}{8} \). It might be noted that for the three triples (1, 5, 5), (3, 3, 5), and (3, 4, 4), the repeated digits are not distinguishable and therefore, there are fewer triples that can be determined from these.

A similar approach was used to find the probabilities for 4-tuples, 5-tuples, and continuing all the ways up to 11-tuples. The following table shows the resulting probabilities.

From the probabilities shown in the table a solution to the problem can be found. The greatest probability is when the student selects three dice. The second best option is for the student to select four dice. The third highest probability will result when a student selects just two dice.

My second approach to solving this problem involved the use of combination numbers. I first designed a table like the one shown on the next page. The values in the table represent the number of ways that a sum can be obtained for a given number of dice. For example, in tossing 3 dice there are 10 triples whose sum is 6. 10 is the underlined value in the table on the next page.
(Table 3)

<table>
<thead>
<tr>
<th>Number of Dice</th>
<th>Sum 2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
</tr>
</tbody>
</table>

\[ x \binom{10}{r} = \binom{x-1}{1} C_1 + \binom{x-1}{2} C_2 + \binom{x-1}{3} C_3 + \binom{x-1}{4} C_4 \]

I noticed that the values shown in each column represented a diagonal in Pascal’s Triangle. As shown in the diagram below, each number in Pascal’s Triangle is a combination number.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

From the diagram, \( \binom{2}{2} = 1 \), \( \binom{3}{2} = 3 \), \( \binom{4}{2} = 6 \), and \( \binom{5}{2} = 10 \). If the pattern continues then the sum of 11 on 3 dice should be \( \binom{10}{2} = 45 \). However, the answer to this problem as shown earlier was 27. Why did the pattern fail? The reason is that in counting the triples using this combination approach, the four triples \((7,1,3), (7,2,2), (8,2,1), \) and \((9,1,1)\) are included. Also all permutations of these four triples are included in the count of 45. There are eighteen triples that are counted that either contain a 7, 8, or 9. When rolling dice the largest integer value is 6. If these 18 triples are subtracted, then the result is 27 as was shown earlier.

I saw the following pattern for this problem:

\[
\binom{10}{2} - 18 = 27 \\
\binom{10}{2} - 3(6) = 27 \\
\binom{10}{2} - 3(4\binom{2}{2}) = 27
\]

In investigating for other cases, I noticed the following pattern. The results are summarized in the table below. Note that \( \binom{x}{r} = 0 \) for \( n < r \).

<table>
<thead>
<tr>
<th>Number of Dice</th>
<th>Probability (Sum=11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \frac{10C_1 - 2(\binom{4}{1})}{6^2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{10C_2 - 3(\binom{4}{2})}{6^3} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{10C_3 - 4(\binom{4}{3})}{6^4} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>11</td>
<td>( \frac{10C_{10} - 11(\binom{4}{10})}{6^{11}} )</td>
</tr>
</tbody>
</table>

Using the table I was able to generalize and write the following:

If \( n \) is the number of dice (2 – 11) and the sum is eleven then the desired probabilities can be found using the following equation:

\[
\text{Probability (sum=11)} = \frac{10C_{n-1} - n(\binom{4}{n-1})}{6^n}
\]

From this generalization I then realized that the list capability of the TI-84 calculator could be used to calculate and display these ten probabilities. The steps necessary to enter this into the TI-84 are outlined on the next page.
In List 1 \((L_1)\) enter the values \((2, 3, 4, \ldots, 11)\).

Set: \(L_2 = \binom{10}{L_1 - 1}, \quad L_3 = L_1 \cdot \binom{4}{L_1 - 1}, \quad L_4 = L_2 - L_3, \quad L_5 = L_4 / 6^{L_1}\)

The values that are stored in \(L_5\) are the set of ten probabilities for this problem. Below is a list of the first seven values that are stored in the lists \((L_1 – L_5)\).

<table>
<thead>
<tr>
<th>(L_1)</th>
<th>(L_2)</th>
<th>(L_3)</th>
<th>(L_4)</th>
<th>(L_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0.5556</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>18</td>
<td>27</td>
<td>1.125</td>
</tr>
<tr>
<td>4</td>
<td>120</td>
<td>18</td>
<td>104</td>
<td>0.08025</td>
</tr>
<tr>
<td>5</td>
<td>210</td>
<td>5</td>
<td>205</td>
<td>0.02826</td>
</tr>
<tr>
<td>6</td>
<td>240</td>
<td>0</td>
<td>240</td>
<td>0.00664</td>
</tr>
<tr>
<td>7</td>
<td>120</td>
<td>0</td>
<td>120</td>
<td>0.015</td>
</tr>
</tbody>
</table>

\(L_{10} = 2\)

To display a scatter plot of these ten probabilities, go to Stat plot and select the following on the Plot1 window.

- Type: \(\bullet\)
- XLIST: \(L_1\)
- YLIST: \(L_5\)
- WINDOW:
  - \(X_{\text{MIN}} = -1\)
  - \(X_{\text{MAX}} = 12\)
  - \(Y_{\text{MIN}} = -0.92\)
  - \(Y_{\text{MAX}} = 0.15\)
  - \(X_{\text{SCALE}} = 1\)
  - \(Y_{\text{SCALE}} = 2\)
  - \(X_{\text{RES}} = 1\)

Set the window to the values shown above and then select the graph option to see a scatter plot of these probabilities. The x-axis represents the 10 values \((2, 3, 4, \ldots, 11)\) and the y-axis represents the probabilities. The trace feature can be used to view these as ordered pairs. The scatter plot is shown below.

I began to wonder if this type of generalization would work for sums other than eleven. It was my hope that given \(n\) six-face dice and a sum of \(x\) where \((7 \leq x \leq 11)\) that the following formula would be true.

\[
\text{Probability (sum} = x) = \frac{x^{-1} \binom{n}{x-1} n \binom{n}{x-5}}{6^n}
\]

Unfortunately, this formula did not work in general. For example consider the case of tossing 3 dice and attempting to obtain a sum of 17. The formula yields the following:

\[
\text{Probability (sum} = 17) = \frac{10C_2 - 3 \binom{10}{3} C_2}{6^3} = \frac{120 - 135}{216} = -\frac{15}{216}
\]

Since the probability of an event cannot be a negative value, it is obvious that this formula does not produce the correct result. There are only three triples \((5, 6, 6), (6, 5, 6),\) and \((6, 6, 5)\) that have the sum of seventeen. The probability is \(3/216\) or \(1/72\).

In finishing with this problem, I wanted to see if I could write a probability distribution for the sum of two dice, three dice, four dice, and five dice. In writing this distribution I used the fact that these probability distributions are symmetrical about the mean. For a single die the mean is 3.5, and for \(n\) dice the mean is \(3.5n\).

I also noticed that for \(n\) dice the first 6 probabilities ranging from \(n\) to \(n+5\) can always be found using the formula

\[
P(\text{sum} = x) = \frac{x^{-1} \binom{n}{x-1} 6^n}{6^n}
\]

where \(n\) is the number of dice and \((n \leq x \leq n+5)\). Looking back at (Table 3) this means that the combination expressions shown in the row where the sum is \(x\) only work for the first six values in each column. For three dice this means that only sums of 3, 4, 5, 6, 7, and 8 can be used with this formula.
The probability distribution functions are shown below and it should always be assumed that $x$ is an integer value.

Two Dice

$$P(\text{sum} = x) = \begin{cases} \frac{x-1}{6^2} & 2 \leq x \leq 7 \\ P(14 - x) & 8 \leq x \leq 12 \end{cases}$$

Three Dice

$$P(\text{sum} = x) = \begin{cases} \frac{x-1}{6^3} & 3 \leq x \leq 8 \\ x - 3 \frac{(x-7)}{6^3} & 9 \leq x \leq 10 \\ P(21 - x) & 11 \leq x \leq 18 \end{cases}$$

Four Dice

$$P(\text{sum} = x) = \begin{cases} \frac{x-1}{6^4} & 4 \leq x \leq 9 \\ x - 4 \frac{(x-9)}{6^4} & 10 \leq x \leq 14 \\ P(28 - x) & 15 \leq x \leq 24 \end{cases}$$

Five Dice

$$P(\text{sum} = x) = \begin{cases} \frac{x-1}{6^5} & 5 \leq x \leq 10 \\ x - 5 \frac{(x-10)}{6^5} & 11 \leq x \leq 16 \\ \frac{780}{6^5} & x = 17 \\ P(35 - x) & 18 \leq x \leq 30 \end{cases}$$

The big surprise was for the case of 5 dice and a sum of 17. Using the formula $P(\text{sum}=17) = \frac{x-1}{6^5} - 5 \frac{(x-10)}{6^5}$, the result is 770 / $6^5$. The actual answer is 780 / $6^5$. At this point I wanted to be sure my calculations were correct. I sent a copy of my calculations together with a manuscript to Dane Camp.

The following is what Dane discovered while verifying my calculations.

When I got the draft of the manuscript in the mail, I was excited. Bill had introduced me to the “What If Not?” technique years before and I was looking forward to seeing another interesting example. Though I enjoyed reading as the application unfolded, my heart sank when I finished and looked at the pages attached, which included detailed computations enumerating each possibility. I am not a great computational proofreader and didn’t look forward to hours of plug and chug. Finally, while attempting to check the work on these sheets, I stumbled across what I thought was a generalization. Had I read this manuscript at some other time, I doubt if I would have noticed it but serendipity intervened. We had just recently finished doing combinatorial analysis in my honors precalculus class and I still had “ball and urn” problems and the principle of inclusion/exclusion running through my head. As I progressed through the sheets, I gradually refined a formula using these two principles.

Using Bill’s narrative as inspiration, I had a leg up on hunting for a generalization. I immediately thought of a ball and urn problem (I think the official parlance is partition, but I’m not sure.) So the question was: How many ways can I distribute $s$ balls (dots) into $d$ urns (upturned faces on the dice) where each urn has at least one ball and at most $6$ balls?

For example, consider the case of rolling a sum of 17 using 5 dice. Each die must have at least one dot showing, so there are $17 - 5$ dots left to distribute to the 5 dice.
Thus we can think of this as a “word” problem where we are looking for the number of ways to rearrange 17 – 5 0’s and 5 – 1 separators. One such arrangement would be 00000/00000/00000 – representing 4, 6, 3, 3, and 1 (the bold zeros are “glued” to insure that there is at least one dot showing on each die). So far, we get the total number of ways to get the sum to be $17 - 5(0) + 5(1) = 1820$. So, in the general case we would glue one dot onto each of the $d$ faces showing. For a sum of $s$, there are $s-d$ dots remaining to distribute. I think of this as what some call a “MISSISSIPPI” problem with $s-d$ 0’s and $d-1$’s. Thus the total, so far, will be 

\[ \frac{(s-d) + (d-1)}{(s-d)} = \frac{(s-1)}{(s-d)} \]

However, we have to remove the cases where more than 6 dots are showing on any particular die, because these are not allowed. For a sum of 17 with 5 dice, we have included cases that are not physically possible, like 0/0000000/00/00000. We must remove the cases where dice are allowed to have more than six dots. So, we glue one dot onto each up turned face, leaving 17 – 5, and then an additional 6 dots onto any one of the 5 up turned faces, resulting in 17 – 5 – 6 to distribute to the rest. Because this situation occurs for each one of the 5 possible choices for the die that gets the extra six dots, we have

\[ \frac{(17 - 5 - 6) + (5 - 1)}{(17 - 5 - 6)} = \frac{10}{6} = 1050. \]

Hence, removing these disallowed cases from the total we derived before, we get 1820 – 1050 = 770. In general, there are $d$ dice where more than 6 dots can occur, each already has one dot and any violator will have at least 6 more. So, we glue those six extra dots onto a face, thus there are $s - d - 6$ 0’s left to distribute to the (still) $d - 1$’s. In other words we must remove

\[ d \left( \frac{(s-d-6) + (d-1)}{(s-d-6)} \right) = d \left( \frac{s-6-1}{s-d-6} \right) \]

from the previous total.

Here is where the principle of inclusion/exclusion kicks into the problem. For some instances there are cases that are double counted. In these scenarios, for example, it is possible to have more than six dots on one face and more than six on another. We have deducted this twice. This explains the discrepancy that Bill discovered while considering 5 dice summing to 17, the arrangement 0000000/000000/0/0/0 was actually deducted twice, once for violating having over six dots on the first die and once for violating having more than 6 dots on the second die. Since it was only supposed to be deducted once, we have to add it back one time. This error is repeated for every pairing of the five dice, so we must add back to the sum an amount equivalent to

\[ \left( \frac{5}{2} \right) \left( 17 - 5 - 2 \cdot 6 + (5 - 1) \right) = \left( \frac{5}{2} \right) \left( 4 \right) = 10. \]

Thus the grand total is 1820 - 1050 + 10 = 780--just as Bill had noted when he was checking his result. In general, we must add back all of the possible pairings of $d$ dice taken two at a time where two of them have more than 6:

\[ \left( \frac{d}{2} \right) \left( \frac{(s-d-2 \cdot 6) + (d-1)}{(s-d-2 \cdot 6)} \right) = \left( \frac{s-2 \cdot 6-1}{s-d-2 \cdot 6} \right). \]

Naturally, when the sum is large enough, we must also consider triple overlap, quadruple overlap, etc. Using the principle of inclusion/exclusion, we know that the terms alternate between being added and subtracted. Since there are $d$ dice, we only need consider the sum up to $d$. (Luckily, the combinations come out to be zero when the overlap is empty.) Putting all of this together we get,
\[
\sum_{t=0}^{d} \left(\frac{d}{t}\right) \left(\frac{s - 6t - 1}{s - d - 6t}\right),
\]
the formula that I used to check out all of Bill’s numbers.

But, even after I sent the manuscript back to him, this rich problem haunted me. Eventually, I noticed two other refinements. First, instead of using \(d\) as the upper bound on the summation parameter, it would be more efficient to employ the floor function \(\left\lfloor \frac{s-d}{6} \right\rfloor\). So, for example, a sum of 17 on 5 dice means that the maximum value on the summation is \(t = 2\) not 5 (as we noted before) since any overlap beyond the double overlap is empty. Also, in the spirit of the “What If Not?” technique, it struck me that there is nothing sacred about cubical dice. If we stick to platonic solids, we just need to replace any 6’s with the appropriate number of faces. So for a tetrahedron, we could simply replace the 6’s with 4’s. In general, if we replace 6 with \(f\), the number of faces of the dice, we get a wonderfully compact formula:

\[
\sum_{t=0}^{f} (-1)^t \left(\frac{d}{t}\right) \left(\frac{s - f\cdot t - 1}{s - d - f\cdot t}\right).
\]

I can’t remember the last time I had so much fun tinkering with a single problem!

We now return to Bill’s account of the problem.

Using the generalization that Dane discovered, I was able to write the program DICESUM for the TI-84 calculator. This program will allow the user to enter the number of 6-faced dice and the sum that is desired. The program will then calculate the corresponding probability. This program can be easily modified so that the user could enter the number of desired faces on each die. The code for the program DICESUM is shown next.

```plaintext
PROGRAM : DICESUM
:ClnHome
:Disp “ENTER THE NUMBER”
:Disp “OF DICE”
:Input D
:Lbl 1
:Disp “ENTER THE SUM”
:Input S
:If (S<D) or (S>6*D)
:Then
:Goto 1
:Else
:0-> W
:0-> T
:Lbl 2
:S-(6*T)-1-> A
:S-D-(6*T)-> B
:(D \_nCr T \_)* (A \_nCr B \_)* (-1)^T-> K
:W+K -> W
:T+1 -> T
:If T ≤ int((S-D)/6)
:Then
:Goto 2
:Else
:6^D-> M
:W/M-> X
:Disp “SUCCESS NUM”
:Disp W
:Disp “TOTAL NUM”
:Disp M
:Disp “PROB “
:Disp X
:End
```

An example execution on the program is shown below. If the user attempts to enter a sum that is impossible to attain, the program will prompt the user with “ENTER THE SUM”. For very large input values of course there is the possibility of overflow.

```
ENTER THE NUMBER OF DICE 25
ENTER THE SUM 17
SUCCESS NUM 780
TOTAL NUM 7776
PROB .100308642
```

---

*Illinois Mathematics Teacher* – Spring, 2008..................................................................................................................22
As this story draws to a close there are a couple of points that I would like to make. The calculations needed to explore this problem did not unfold in a single day. I worked on this problem on and off for several months. A problem of this type can take a great deal of time to investigate and refine. Students should be aware that every math problem does not need to be solved in a short period of time.

I want to emphasize how teamwork was the recipe that eventually led to the discovery of the generalization. I would not have found the generalization on my own. I doubt that Dane would have investigated a problem of this type without seeing my initial calculations. There were some exciting moments as we traded emails regarding the construction of this manuscript.

There are still several unanswered questions. The first three of the six attributes were discussed during the course of this article. However, there are three that remain as challenges to the reader. For the computer programmer there are several possibilities for constructing programs that involve probabilities related to the sum of dice. To statistics teachers there are topics and notation that can be incorporated into lessons and shared with students.

In closing I give credit to the “What if not?” technique for helping this story to unfold. This technique provides an excellent framework for generating problems and promoting mathematical investigations. For this reason this strategy must be shared with students at all grade levels. I believe that many significant discoveries can be made by students who are exposed to the “What if Not?” technique.

Your presence is requested at the 59th Annual Meeting and Conference of the Illinois Council of Teachers of Mathematics, “Connections in the Math Landscape”. It will be held October 16-18, 2008 in Peoria, Illinois. We hope to see you there.
A Fun Polygon Activity

Cindy Budzikowski, -Wheaton North High School, cbudziko@cusd200.org

One of my favorite challenge problems that I have used in my Geometry classes every year involves regular polygons - lots of regular polygons! This challenge originated from my student teaching 14 years ago with a simple problem requiring knowledge of regular polygon angle measures. The original problem came from a textbook, the name of which I unfortunately do not recall. The problem, however, I will never forget! Here was how it started.

Text book Problem:
Given regular pentagons ADEBH and BFCGI and equilateral triangle HBI, find m<ABC.

This problem provided some good practice on the interior angle measures of a regular polygon. Well, I figured we could alter this problem a little and get some more practice with these regular polygon angle relationships. As a class, we changed the problem slightly.

Modified Problem:
Given regular squares ADBH and BFCG and equilateral triangle HBG, find m<ABC.

Some interesting things happen as you add sides to the polygons. At some point, <ABC becomes a straight angle. Then you will see that the gap between the two polygons shrinks and the polygons eventually share a side. Add more sides and the polygons begin to overlap and the real challenge begins. Have some fun with this extended problem. I usually take a couple of days on this activity with my classes. I’ve used the following extended polygon activity for both advanced and intermediate level students. Enjoy!

Note: The solutions for this activity can be found on page 31.
# Geometry – Polygon Problem

<table>
<thead>
<tr>
<th># of sides</th>
<th>m&lt;ABC</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
</tr>
<tr>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>

Work for generalization and restrictions on n:
1. Is there a time when the two regular polygons will share a side? If so, how many sides will each regular polygon have?

2. When the polygons start to overlap, as the 14 sided regular polygons do below, an angle is formed by two overlapping sides of the polygons and one vertex of the triangle (<XYZ). Find an equation that can be used to determine the m<XYZ formed by two overlapping n-sided polygons. This equation should be written in terms of n.
Making Content Relevant to Students: iPod Playlists and Permutations

Neil Eckman, Sandwich High School, neil.eckman@gmail.com

Introduction

Though my experience with teaching is still in its infant stages, I have found that the problem with getting students motivated for mathematics starts with making it relevant to them. Educators across the country are currently in competition, not necessarily against each other, but against the wave of technology that students use in their daily lives. The most recent phenomenon is unquestionably the iPod Mp3 player. Music provides an outlet and is an important part of most of our lives, and it is particularly important in the lives of our youth. It is our students who use these devices on a daily basis, on their way to school, before school, between classes, etc. Simply put, the cultural revolution of the iPod has created a deep impact in this generation of music lovers. So, why not use this phenomenon to build student interest?

Shuffle, a Randomizer and Personal Arranger

One of the wonderful features of the iPod is its versatility. Its users can manually create playlists on their computer from their library on iTunes - the software that enables songs to be downloaded to the iPod. With the playlists, users can select to play songs in orders by Song Name, Artist, Album, Genre, etc. There is also the option of shuffle, a feature that randomly selects and orders songs from a playlist based on the criteria that is selected to be randomized (song, artist, album, genre). Using shuffle as my randomizer, I created a discovery activity that will help students grasp the basic concepts of permutations and distinguishable permutations.

Distinguishable Permutations occur when an element of a set (letters, numbers) is repeated. For instance: the distinguishable permutations of EYE totals 3 arrangements (EYE, YEE, and EEY). However, if the E’s are considered to be different elements, then there would be 6 ordered arrangements (EYE, EYE, YEE, YEE, EEY, and EEY). In order to calculate the number of distinguishable permutations you need to divide nPr by the number of times an element is repeated. In our example of the permutations of EYE taken 3 letters at a time, the calculation is as follows: \( \frac{3!}{2} = \frac{6}{2} = 3 \) ways. Two represents the number of times “E” was repeated.

(Note: the outline of the activity was adapted from the explore activity from section 6-1B of Dossey & Vander Embse et al., Focus On Advanced Algebra: An Integrated Approach, pg. 428, 1998, Addison Wesley.)

Prior to this Lesson

Students will need to know the multiplication counting principle in order to do the calculations. Also students should have knowledge of factorials.

Before Handing out the Activity, Build Interest…..

Take a survey of your students asking:

1. How many of you own iPods?
2. How many of you use the shuffle feature?
3. What do you think of the shuffle feature? Is it as random as you think it should be?

Directions

You will not need an iPod to do the lesson, though it would be a nice touch. In the example, choose a lower number of songs for your playlist, like 10, so that students’ calculators do not overflow. Most students should know how the shuffle feature works, but just in case they do not, make sure to explain that the shuffle feature will randomly select and order songs. “Order/Arrange” should be the operative words that you communicate to them. Allow about 10 minutes for students to work on the worksheet, walk around the classroom and answer questions, but do not give away answers, allow them to form their own ideas and guide them.

After Completing the Worksheet

Have a discussion with your class over each of the problems and begin to deliver instruction on permutations and distinguishable permutations. Possibly introduce the formulas and have students calculate 10P10, 10P2, and 10P3 on their calculators. Refer back to the activity and make connections with each topic with the calculations between the multiplication counting principle method and the formula. Students will then be able to reflect on their thinking process during the activity and make connections to the new content that is being presented.

Possible Extensions

With the particular Advanced Algebra class I taught, we did not derive the formula for permutations or distinguishable permutations, but in an Honors section, Pre-Calculus or other advanced topic classroom one could extend this activity into that realm. You could also turn this into a project/experiment for a Probability and Statistics class where you could calculate probabilities and test for randomness in terms of song name, artist, album, and genre.

Thank You’s

I would also like to thank my cooperating teacher at Naperville Central, Jeff Danbom, who helped me refine this activity before I presented it. I would also like to thank Andy Samide, teacher at Montini Catholic High School and Wheaton North High School for always pushing me to be creative and to submit this as an article to ICTM.

Note: The solutions for this activity can be found on page 33.

References

iPOD: Shufflin’ with Permutations!!!

1. An iPOD on Shuffle randomly selects songs from a playlist that the user has created. If a playlist was created and contained 10 different songs, how many ways could the songs be played without repeating a song?

2. Using the 10 song playlist from part (a). Say you only wanted to listen to only two songs on the playlist, how many ways could two songs be played?

3. How many ways could only three songs be played?

4. Two different songs from two different artists are included in a four song playlist. For example, lets use two songs from Weezer (W) and two songs from the Beatles(B). Answer the following questions.
   a. List the possible orders by artist. (ex. BBWW is one)
   b. How many possible orders by artist are there?

5. Using part (4). With all else being equal, what is the probability that two songs from the same artist are played as the first two songs of the playlist?
Watson and the Mysterious Number One

Andy Samide, Montini Catholic High School, asamide@montini.org

Clearly the number one could be used because the width of the rectangle would have a length less than the length of 4. But, if 2 were used instead of 1, the result would also give the width of the rectangle a length less than 4. So, how did Watson deduce that 1 would be the correct number to use for the problem? That is the mystery that puzzled the class and even Watson when he was asked to justify his use of the number one.

Intrigued? Puzzled? Confused? Then read on for a delightful problem for you and your students to solve.

Let’s start at the beginning. The problem—a rectangle 5 units in length is folded so that the opposite vertices coincide and the length of the fold is $6$. Determine the width of the rectangle if the width is to be less than 4—was first given to two advanced geometry classes at Wheaton North High School. It was given to Watson who is in the ninth hour honors geometry class at Montini Catholic High School several days later. The problem was assigned to Watson because he was not very attentive to the discussion of the lesson of the day. And he really enjoys being challenged.

It took several weeks before Watson was called upon to share his solution to the problem which he was assigned. And that is when he produced his number one for the length of a leg of one of the right triangles he drew in his diagram.

Intrigued by Watson’s 1, the ninth hour class justified that Watson’s 1 was correct as shown in the following diagram.

Thus, if 1 was replaced with 2, as mentioned previously, the width of the rectangle would be less than 4; however the length of the rectangle would not be 5.

Once the class established and were convinced that Watson’s 1 was correct, the curiosity of the class continued when a student asked the following question:

If the length of the fold has as its radicand and integer that is one more than the integer value of the length of the rectangle, then is the length of the width the square root of the length of the rectangle?

“Real math is not crunching numbers but contemplating them and the mystery of their connections” — Unknown

If the answer to the above question is correct, must the length of the leg of the right triangle shown in the first diagram always be Watson’s 1?
In previous articles that have been submitted, complete worked out solutions were always included. Since the classes have not solved the present problem, a solution is not, unfortunately, included. And, by the way, Watson is still trying to remember why he used the number 1.

ENJOY!

NOTE: Since writing this article, a solution has been found. It can be found on page 46.

A Fun Polygon Activity – Solutions
(from the article beginning on page 24)

<table>
<thead>
<tr>
<th># of sides</th>
<th>$m\angle ABC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>150°</td>
</tr>
<tr>
<td>5</td>
<td>168°</td>
</tr>
<tr>
<td>6</td>
<td>180°</td>
</tr>
<tr>
<td>7</td>
<td>171.43°</td>
</tr>
<tr>
<td>8</td>
<td>165°</td>
</tr>
<tr>
<td>9</td>
<td>160°</td>
</tr>
<tr>
<td>10</td>
<td>152°</td>
</tr>
<tr>
<td>14</td>
<td>145.71°</td>
</tr>
<tr>
<td>20</td>
<td>138°</td>
</tr>
<tr>
<td>25</td>
<td>134.4°</td>
</tr>
<tr>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>

Work for generalization and restrictions on $n$:

For $3 < n \leq 6$, $m\angle ABC = 240 - \frac{360}{n}$

For $n > 6$, $m\angle ABC = 120 + \frac{360}{n}$

1. yes, 12 sides

2. $m\angle XYZ = \left(60 - \frac{720}{n}\right)^\circ$
In teaching statistical processes, it is important that there be application to real-world settings and activities. When this is done, students are more likely to see the meaning of the steps being developed.

One such activity involves using the Chi-Square statistical test and its applications to counting Milk Chocolate M&M’s of different colors. Most students are aware that these M&M’s chocolate candies come in six different colors: orange, green, red, yellow, brown, and blue.

According to the information provided by Mars Incorporated, the manufacturer of M&M’s, the following should be the color distribution for the Milk Chocolate M&M’s:

<table>
<thead>
<tr>
<th>COLOR</th>
<th>EXPECTED NUMBER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orange</td>
<td>20% of 365 = 73</td>
</tr>
<tr>
<td>Green</td>
<td>16% of 365 = 58.4</td>
</tr>
<tr>
<td>Red</td>
<td>13% of 365 = 47.45</td>
</tr>
<tr>
<td>Yellow</td>
<td>14% of 365 = 51.1</td>
</tr>
<tr>
<td>Brown</td>
<td>13% of 365 = 47.45</td>
</tr>
<tr>
<td>Blue</td>
<td>24% of 365 = 87.6</td>
</tr>
</tbody>
</table>

To test the Null Hypothesis, we shall use the Chi-Square statistic. Let us construct Table 1 with column entries as follows:

\[
\frac{(O-E)^2}{E} = \text{A measure of the discrepancy between O and E.}
\]

<table>
<thead>
<tr>
<th>COLOR</th>
<th>O</th>
<th>E</th>
<th>((O-E)^2/E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orange</td>
<td>66</td>
<td>73</td>
<td>0.67</td>
</tr>
<tr>
<td>Green</td>
<td>71</td>
<td>58.4</td>
<td>2.72</td>
</tr>
<tr>
<td>Red</td>
<td>51</td>
<td>47.45</td>
<td>0.27</td>
</tr>
<tr>
<td>Yellow</td>
<td>65</td>
<td>51.1</td>
<td>3.78</td>
</tr>
<tr>
<td>Brown</td>
<td>46</td>
<td>47.45</td>
<td>0.04</td>
</tr>
<tr>
<td>Blue</td>
<td>66</td>
<td>87.6</td>
<td>5.33</td>
</tr>
<tr>
<td>TOTAL</td>
<td>365</td>
<td>365</td>
<td>12.81</td>
</tr>
</tbody>
</table>

According to the “theoretical” numbers supplied by Mars, the expected color distribution should be as follows:
that O and E are relatively far apart, as is the case for blue.

The sum of this discrepancy column, 12.81, is called the Computed Chi-Square Statistic (CCSS). A determination must be made as to whether the CCSS is large enough to cause us to reject the Null Hypothesis. To make this decision a “referee” is needed. This referee is found in the Table Chi-Square Statistic (TCSS).

To read a Chi-Square table, the degrees of freedom must first be determined; that is, one less than the number of categories (colors). In our case, the degrees of freedom is 6 - 1 = 5. This means that if the total number of M&M’s were known, and the number in each of five categories were known, the number in the sixth category could be calculated.

The significance level is the probability of rejecting a Null Hypothesis which is in fact true. This could occur because the sample is not representative of the population. From a Chi-Square table, we find:

<table>
<thead>
<tr>
<th>SIGNIFICANCE LEVEL</th>
<th>TCSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>9.236</td>
</tr>
<tr>
<td>5%</td>
<td>11.070</td>
</tr>
<tr>
<td>1%</td>
<td>15.085</td>
</tr>
</tbody>
</table>

The decision mechanism for the null hypothesis is:
- If CCSS > TCSS, then CCSS is large in the “judgment of the referee.” If this is true, reject the Null Hypothesis.
- If CCSS < TCSS, then CCSS is small in the “judgment of the referee.” If this is true, accept the Null Hypothesis.

Our CCSS of 12.81 is larger than the TCSS’s for SL = 10% or 5%, but smaller than the TCSS for SL = 1%. In other words, there is sufficient evidence to reject the Null Hypothesis for the first two significance levels, but not enough evidence to reject the Null Hypothesis at the more stringent 1% significance level.

Challenge for readers and their students: Replicate this experiment for other large bags of Milk Chocolate M&M’s. Find (on the “mms.com” website) the color distribution for other types of M&M’s and test the hypothesis suggested by this site with “real” data.

Making Content Relevant to Students: iPod Playlists and Permutations

Solutions
(from the article beginning on page 27)
1. $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3,268,000$ ways or $10! = 3,268,000$ ways  
2. $10 \cdot 9 = 90$ ways  
3. $10 \cdot 9 \cdot 8 = 720$ ways  
4. a. WWBB BBWW  
   WBWB BWBW  
   WBBW BWWB  
   b. 6 possible orders  
5. $\frac{2}{6} = \frac{1}{3}$
An Introductory Experience with Lesson Study

Mary T. McMahon, North Central College, mtmcmahon@noctrl.edu

Authors’ note: A previous version of this paper has appeared in Issue 4 of Success in High-Need Schools, the online journal of Associated Colleges of Illinois’ Center for Success in High-Need Schools.

Abstract

This paper is a report of a summer lesson study experience with middle school mathematics and science teachers at Cowherd Middle School in East Aurora School District 131. Teachers planned, taught, evaluated, revised, retaught and reflected on a lesson concerning spatial visualization and geometric reasoning. Teachers grew in their understanding that the lesson study process is valuable for their preparation of challenging lessons and that the process supports their professional development as teachers.

An Introductory Experience with Lesson Study

In June 2001 at an Illinois Math Teacher Educator Meeting, Claran Einfeldt, formerly a Division Administrator with the ISBE Division of Mathematics and Science, gave all participants a copy of The Teaching Gap (Stigler & Hiebert, 2000). Ideas of a teacher-led professional development called Lesson Study were fostered in the text. The original ideas came from Japan where teachers routinely participated in lesson study. I am submitting this report to offer assistance to others who may be interested in undertaking lesson study.

Synopsis of Japanese Lesson Study

In Japanese lesson study, teachers come together to decide on a research theme or main goal of the lesson (Lewis, 2002). This is a statement of the broad aim of the lesson study that might be aligned with the school’s mission statement. Next a subject area is chosen and then the teachers choose the topic and the unit/lesson goals. The teachers are now ready to research the topic and plan the lesson. In the planning discussion, teachers focus on student learning. They decide on a list of carefully worded questions and attempt to anticipate what students’ responses will be to the teacher questions and classroom activities. During this planning stage, teachers question their beliefs, reflect on why they teach as they do, learn from their colleagues and build a stronger collegial network. The next step would be for one member of the group to teach the lesson and for the others to observe the students and their reactions to the lesson. Soon after the lesson, the teachers would meet to discuss the lesson and make revisions based on student response. Ideally a different member of the group would teach the revised lesson while the others would observe student response.

“Best Practice”

Lesson study employs “Best Practice” because the teachers are being truly collaborative. They research the topic, discuss and question others’ opinions and suggestions and are involved in an in-depth systematic examination of their practice as their focus shifts away from what is
presented to what is learned. Teachers develop habits of mind of research to investigate teaching and learning, communication of new ideas and approaches and self-efficacy as they realize that they can make a difference in student learning. (Stepanek, J., 2007) “Because lesson study is carried out in classrooms, the problem of applying the [research] findings to classrooms disappears.” (Stigler, p. 165).

The benefits for students are that the lesson study lessons are very engaging. They are inquiry-based, have input from many teachers and target specific learning needs. The classroom is learner centered with students learning from classmates that there are multiple ways to a problem solving solution while explaining their thought process orally and in writing. Teachers tend to question by scaffolding.

The teacher benefits include encouragement for long term professional development as they plan more engaging lessons and gain insights from other teachers. They become more reflective teachers with immediate feedback and gain greater understanding of their colleagues. Also they have an opportunity to discuss whole school issues while planning.

Background

In winter term 2004, a colleague and I engaged my pre-service practicum students in a modified form of lesson study. In fall of 2005, we discussed how we could undertake lesson study with in-service teachers. I approached Joan Glotzbach, principal of Cowherd Middle School in East Aurora School District 131, where I had previously supervised preservice teachers for observation experiences, and asked if she would be interested in starting a lesson study group. Joan discussed it with teachers in her building and three agreed to participate.

The Planning Process

In early April 2006, we scheduled a meeting with Joan and three classroom teachers to introduce our plans and answer their questions. The teachers were interested and offered to recruit more teachers.

We decided to offer the summer professional development institute for three afternoons. We emailed the teachers to further acquaint them with our professional backgrounds and the purpose of lesson study. We also gave them calendar options and a choice of dates. One of the choices happened to be during the last week of summer school. The teachers selected that option and it was decided to hold the sessions on Wednesday and Thursday from 12-5 and Friday from 7:15-3pm that week with lunch provided at all sessions. With this option we would be able to teach the lesson in two summer school classes.

In early May I attended a Lesson Study conference. There several participants talked about a video that they had found very helpful in introducing teachers to lesson study. It is entitled To Open a Cube (Lewis, 2001) and information and clips may be found at http://lessonresearch.net/opencube.html. We decided that it would be a good basis for the teachers’ initiation into lesson study. I also invited Michelle Pope, one of my former students, who is a seventh year teacher and lesson study participant at Stevenson High School to give a talk to the teachers about her experiences with lesson study. Michelle has co-authored a chapter in Teachers Engaged in Research: Inquiry in Mathematics Classrooms, Grades 9-12 (Teachers Engaged in Research), and many of her remarks were based on her experiences that she had related in the chapter on lesson study.
To prepare for the institute, we made up a notebook for each participant with the agenda, articles and web resources. Two days before the start, we had five enrollees. The day before, we received three more. When we arrived at the school for the first meeting, we were amazed to have ten teachers and a numeracy coach in attendance! Five of the teachers taught math. The other five taught science and math lab. We were ecstatic to have so many.

Summer Institute
Day 1 - Introduction

We began at noon with lunch, a request to complete an information sheet and anonymous beliefs questionnaire, a short introduction about our backgrounds, why we chose that school and why we were so interested in lesson study. I then introduced Michelle Pope. Michelle explained how Stevenson H.S. facilitates lesson study and her involvement. She introduced the steps of The Lesson Study Cycle adapted from (Carter 2006):

• Set goal,
• Conduct research,
• Plan a lesson,
• Teach and observe the lesson,
• Evaluate the lesson, and Reflect,
• Revise and repeat.

Her power point included clips of teacher lesson study planning discussions and actual lessons that had been taught with a lesson study script with teacher/observers in the classroom. Michelle answered many questions about practices used in lesson study and in her classroom. She stressed that the planned lesson was a valuable end product but the process with the teachers interacting and the focus on the learner was even more important.

After a short break, we introduced the activity from *To Open a Cube* (Lewis, 2001). The teachers answered questions and began work on the activity of how many different nets (flat patterns) could be cut from a paper cube. All worked diligently. The table discussions manifested the depth of their interest, their use of problem solving skills and their different learning styles. All began with scissors to “open a cube”. After cutting open one or two cubes, some tried to analyze the problem and worked on grid paper. When the teachers completed the activity, they viewed the DVD entitled *To Open a Cube*. The video is of a public research lesson taught by Dr. Akihiko Takahashi to fifth graders in the San Mateo-Foster City School District, CA. The teachers were able to see how Dr. Takahashi fostered students’ thinking, interactions between the teacher and students, the classroom arrangement, and the students’ participation in the activity. After the video, the teachers were given a reflective writing prompt for homework:

*Please take a few minutes to express your thoughts and initial reactions to the video on implementation of lesson study. How will Cowherd students’ responses be similar to those of the students in the video? How will they be different? Does a lesson study approach to professional development seem possible for Cowherd teachers? Would a lesson study approach to professional development be effective for improving students’ instructional opportunities? (In what ways?)*
Day 2 – Planning the Lesson

The afternoon session again began with lunch and many comments and questions about the previous day’s activities. Some were content related: How many different nets were possible. Was there a formula to determine the number? Others were about the behavior of the teacher and students in the video: Would their students react in a similar manner?

We were now ready to plan the lesson. The teachers divided into two groups: math and science. This was done so that everyone would have a voice in the planning stage. We also thought that the two content areas might approach the lesson differently and they did. After each group brainstormed for a half-hour, the two groups came together and discussed their ideas for the lesson. On the previous day questions had arisen about the lesson plan form. Table 1 at the end of the article is an abbreviated form of one from Karen Jacobson (2006) & Teachers from Northside Elementary School, Montevideo, Minnesota, that we used.

During the planning, the teachers debated and questioned most suggestions. They focused on what they thought the student would think about the lesson. For the introduction, students were to be shown a cube and asked what they knew about it. Precise mathematical vocabulary was to be stressed. The teachers decided to use as the “lesson hook”, the idea of a game imprinted on the inside of a cereal box and how important it was to think about how the box should be cut open to avoid damaging the game. They decided then to introduce the concept of net by giving students the following scenario:

Here is the game Thinking inside the Cube. The object of this game is to discover the mystery net. (One randomly chosen by the teacher and hid behind a paper on the board.) You will need to cut along the edges so that the faces are attached and intact. The winner or winners who find the mystery net will get 100 grand. You may work alone or with a partner. Remember there are multiple nets for this cube, but only one net will match the mystery net and will win the 100 grand.

Throughout the planning, the teachers collaborated in an animated discussion. Time was running out but the lesson came together. We asked for volunteers to teach the lesson in two class periods on the following day and immediately two teachers agreed.

Day 3 – Teaching the Lesson

The teachers arrived at 7:15am in preparation for the class starting at 7:30am. It was an 8th grade class on their last day of summer school. How would they react to a different teacher, eleven other adults in the room and an activity to participate in? The lesson was well received. Students participated wholeheartedly. The teacher/observers followed lesson study procedure. They were strictly observers – they did not interact with the students in any way. During the whole group part of the lesson, they stood around the perimeter of the room. During the activity time, they moved around the room and listened to student conversations and observed their activities. They took notes throughout. To the observers’ surprise, they were well received in the classroom and the students worked diligently throughout the lesson.

After the lesson, the teacher of the lesson and the teacher/observers discussed how the lesson was received. The debriefing began by following lesson study protocol Curcio, F. (2002). I thanked the teacher of
the lesson and asked her to comment on the lesson. Then each teacher/observer took a turn by first thanking the teacher of the lesson and making their comments on the lesson. After all participants had a turn to speak, there was discussion on what should be revised. They were pleased overall but decided on a few changes. For the next teaching of the lesson, they decided to have a student cut the cereal box into a net and changed how the winning prize would be distributed. They discussed whether to encourage/discourage students to work in pairs/groups, whether to put all supplies in a central area, and numerous other small changes to the lesson. The second volunteer was then ready to teach the lesson.

The second class was a 7th grade class. They were slower to respond to the lesson but as the lesson progressed, their on-task behavior improved. They also did not appear to be affected by the teacher/observers in the room. Many of the revisions discussed were applied in the lesson. The teacher/observers took notes as they had done in the previous class. One teacher/observer recorded the number of students on task and prepared Table 2 (at the end of the article) comparing changes in on-task behavior as each class session progressed. There was some discussion concerning the differences between the classes. The teachers who were familiar with the students attributed it to the makeup of the classes. Some were surprised at the high level of interest in both classes given that this was the last day of summer school.

The same protocol was used as in the first debriefing. Three main themes came from the comments: How important was it to “stick to the script”, should students be encouraged to work in groups/pairs and how should the supplies be dealt with. The teachers discussed in depth the pros and cons of sticking to the carefully constructed script and examined the pedagogical implications. They seemed to agree that even though much time had been taken to carefully plan that there were times that it was imperative to deviate from the script. No agreement was reached on whether placing the students in pairs/groups would work with their students. The teachers liked the idea of having the students decide which supplies and tools they would need to accomplish the activity even though some students took one of everything available. The discussion was rich and the teachers said that they would like to continue in the fall.

Fall

At the teachers’ suggestion and with the approval of Cowherd’s principal, we continued to meet with the same group of math and science teachers. On the first full day of school we considered overarching goals with a tie-in to the school mission statement and contemplated the ideal student vs. actual student. We discussed how these goals should be reflected in the teachers’ lessons. We also recapped what we had done in the summer. Lesson study meetings were held after faculty and department meetings twice a month from 4:15-7:00pm. Our major difficulty was the interference of after school activities with teachers’ attendance.

The math and science teachers worked on a joint lesson on measurement and taught it in early November. At that point we hoped that the teachers would take ownership and value this form of professional development facilitated by teachers and focused on the learner. There are plans to continue in the summer.
Conclusions

We were very pleased by the teachers’ involvement in their first lesson study cycle. They understand that while they are trying to plan the best lesson possible, they are also engaging in the rich discussion of planning, reflecting and revising of the lesson with the focus on the learner's perspective. In this process they think more deeply about the content and collaborate professionally to improve students' learning opportunities. For example, in the reflective writing, one teacher commented,

* A lesson study approach for Cowherd will be effective for improving student instruction, because the lesson will be focused. It probably will increase student alertness and awareness as we look at student reaction towards the lesson and tweak it so that students will understand.

Another teacher remarked on the final evaluation,

* It was wonderful to plan with my co-teachers. I learned a lot about their teaching styles and what they thought was important to a lesson...As we go through the school year, we will be able to apply these methods to our own classroom which will make this even more worthwhile.

We regard comments such as these as indication of awareness on the part of the teachers of the potential lesson study holds to improve their overall lesson planning skill as well as to improve learning opportunities for students.

References


Karen Jacobson & Teachers from Northside Elementary School. (2006). Tool for Planning and Describing Research Lessons. Aspects of this tool were derived from lesson plans provided by Makoto Yoshida of Global Education Resources, L.L.C (myoshida@globaledresources.com), and by the Greenwich Japanese School, CT. In addition, a number of the planning questions suggested in this document were developed by Dr. Fritz Staub and Lucy West, under the auspices of the Learning Research and Development Center, University of Pittsburgh, and Community School District 2, New York City. Barbrina Ertle, Sonal Chokshi, & Clea Fernandez. ©2001, Lesson Study Research Group (lsrg@columbia.edu).


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**Table 1**

<table>
<thead>
<tr>
<th>Steps of the lesson: learning activities and key questions (and time allocation)</th>
<th>Student activities/expected student reactions or responses</th>
<th>Teacher’s response to student reactions / Things to remember</th>
<th>Goals and Method(s) of evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>This column is usually laid out in order by the parts of the lesson (e.g., launch, investigation, congress, extension/applications, etc.), and also includes the allocation of time for each of these parts. This column should also include a description of key questions or activities that are intended to move the lesson from one point to another.</td>
<td>This column describes what students will be doing during the lesson, and their anticipated reactions or responses to questions/problems you will present.</td>
<td>This column describes things that you want to remember to do/not to do within the lesson as well as other reminders to yourself. Also, as you have anticipated student responses and reactions (previous column), this column provides a place where you can think through how you might use those responses and reactions in synthesizing a true learning experience within your classroom.</td>
<td>This column describes the goals that are being focused upon during each part of the lesson, and for each activity/problem. It should also include a concrete description of how you will determine that you have achieved each of these goals.</td>
</tr>
</tbody>
</table>

**Student Participation during Two Minute Intervals Every Five Minutes**

<table>
<thead>
<tr>
<th>Minutes in Class Period</th>
<th>5</th>
<th>12</th>
<th>19</th>
<th>26</th>
<th>33</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of Students on Task</td>
<td>100</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>

**Table 2**
Thinking Inside The Box

State Goal
9.7.11: Identify a three-dimensional objects from its nets.
9.7.12: Recognize which attributes (such as shape, perimeter, and area) change or don’t change when plane figures are composed, decomposed, or rearranged.

Lesson Goal
To develop problem solving skills by discovering all possible nets of a cube.

Goal of the Research Lesson
To prove they have a different net through communicating their ideas.

Process of the Research Lesson:

<table>
<thead>
<tr>
<th>Steps of the lesson:</th>
<th>Teacher’s Response</th>
<th>Student activities/expected response</th>
<th>Goal and Method(s) of evaluation</th>
</tr>
</thead>
</table>
| Intro of the cube:   | Does anyone know what this object is? | Cube
Excellent! | To have all students agree the object is a cube. |
|                      | *Box/ Dice* Good. Does anyone have another name for this? | *Square*
Show a square-Is this the same as what I have in my hand? |
| What can you tell me about this cube? | 6 faces
12 edges
8 vertices | |
| Cereal Box Game:     | Raise you hand if you have ever seen a cereal box with a game? | Hands raise |
|                      | Where is the game located? | On the back |
|                      | Have any of you ever seen them on the inside? | Yes |
|                      | How would you neatly get to the game if it is inside the cereal box? | Cut it open along the edge |
Introduce the game: Here is the game **Thinking inside the Box.** The object of this game is to discover the mystery design. You will need to cut along the edges so that the faces are attached and intact.

The winner or winners who find the mystery design will get 100 grand. If you work alone you will get the 100 grand to yourself. If you work with another person you will have to share the 100 grand.

| If you have 2 people in your group how much of the 100 grand will each person get? | 50 grand |
| If you have 4 people in your group how much of the 100 grand will each person get? | 25 grand |

Getting Started: As you come up with designs tape them to the green piece of paper. Make sure your design is not already up there and write your name or names by the design. You have 10 minutes to get as many as you can up there.

| What materials will you need to solve this mystery? | Cubes  
Scissors  
Tiles  
Grid Paper |
| Any other supplies that you may need will be on the table. |
| After 10 minutes: If no one has the design: Show the mystery design and how to get to the mystery design. |
| To Conclude: Journal: I saw a lot of good strategies to solve this mystery. I would like you to take the next 5 minutes to write down: How you tried to find the solution? | Students begin to write |
An Application of the Linear Combination Method in Finding the Equations of Lines

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Sudhir Goel, Valdosta State University, sgoel@valdosta.edu

In the high school algebra, as well as in the two-year college curriculum, a standard topic in algebra is writing linear equations. Students usually learn several techniques to do this. Typically included in covering this topic is writing the equation of a line parallel or perpendicular to a given line and passing through a given point. After mastering linear equations students are next taught to solve a system of linear equations.

There are several ways to solve a system of equations. One of the methods is often referred to as the Multiplication-Addition Method. This method is also called the Linear Combination Method by some texts. The goal of this method is to find the intersection point of two lines by adding a multiple of one linear equation to a multiple of a second equation. Once students can solve a system by this method they are ready to tackle the other types of problems.

Another standard topic is finding the equation of a line through the point of intersection of two given lines and a point external to both lines. Most students would solve this problem by finding the point of intersection of the two lines and then writing the equation of the line from the two points that lie on the line. Suppose in posing the problem it is stipulated that students cannot find the point of intersection of the two lines. The remainder of this article will focus on a way to solve this problem under this new condition.

In “Parallel and Perpendicular Lines in a College Algebra Classroom”, Goel and Petrella [2001] studied the routine problem of finding an equation of a line parallel and/or perpendicular to a given line through a point not on the given line. They discussed the conventional method in which the slope of the first line is found, then, using the point-slope formula, the equation of the new line can be determined. The authors then discussed and proved an alternate method for solving this problem. The “new” method is given by the following theorems.

Theorem 1
The equation of a line parallel to a given line \( Ax + By = C \) and passing through a point \((r, s)\) not on the line is \( Ax + By = D \), where \( D = Ar + Bs \).

Theorem 2
The equation of a line perpendicular to a given line \( Ax + By = C \) and passing through a point \((r, s)\) not on the line is \( Bx - Ay = D \), where \( D = Br - As \).

In this paper, we extend the idea presented in Goel and Petrella’s paper to solve the problem posed above. We then prove a theorem to show why the method presented below works. Note that when we find the point of intersection of two lines using the method of linear combination, we add a scalar multiple of the equation of one line to the equation of the other line. In our conjecture we are essentially doing the same thing.

\[
(\text{equation of one line}) + k(\text{equation of the second line}) = 0 \tag{1}
\]
We would like to point out that equation (1) provides an excellent opportunity for students to really see what is meant by linear combination.

Let’s consider the following problem:

Find the equation of a line that passes through the point \((1,1)\) and the point of intersection of the two lines \(5x - 2y = 7\) and \(8x + 3y = 15\).

The conventional way to solve this type of problem involves first finding the point of intersection of the two given lines and then using this point along with the given point to find the equation of the desired line. We first observe that the point \((1,1)\) is external to both lines. Next, we solve the system to obtain the point of intersection \(\left(\frac{51}{31}, \frac{19}{31}\right)\). Now, using the two points, the slope of the desired line can be calculated. This slope is \(-\frac{3}{5}\). Plugging this into the point-slope formula for a line, we obtain the equation of the desired line. This equation is \(3x + 5y = 8\).

Most teachers take the problem a bit further and show the students the solution visually. Bremigan (2001) stated that “Reasoning with visual representations is an important component in solving many mathematical problems and in understanding many mathematical concepts and procedures.” By graphing the given lines and point, the student can gain a better understanding of the problem and solution.

We will show yet another approach to this routine problem. We conjecture that the equation of a line passing through the point of intersection of two lines: \(A_1x + B_1y = C_1\) and \(A_2x + B_2y = C_2\), and a point \((r,s)\) external to both lines is

\[
(A_1x + B_1y - C_1) + k(A_2x + B_2y - C_2) = 0, \tag{2}
\]

where \(k = -\frac{(A_1r + B_1s - C_1)}{(A_1r + B_2s - C_2)}\), i.e. \(k\) is obtained by substituting \(x = r\) and \(y = s\) into equation (2). We will first demonstrate the validity of our conjecture with the help of the numerical example discussed above and then we will prove it in general.

Example. Find the equation of a line that passes through the point \((1,1)\) and the point of intersection of the two lines \(5x - 2y = 7\) and \(8x + 3y = 15\).

Solution: We first observed that the point \((1,1)\) is external to both lines. Also recall that the solution of the system of equations is the point \(\left(\frac{51}{31}, \frac{19}{31}\right)\) and that the standard form of the equation which contains the points \((1,1)\) and \(\left(\frac{51}{31}, \frac{19}{31}\right)\) is \(3x + 5y = 8\).

If we now use the conjecture stated above, then equation (1) becomes

\[
(5x - 2y - 7) + k(8x + 3y - 15) = 0.
\]

To obtain the value of \(k\), we substitute \(x = 1\) and \(y = 1\) in the above equation. This yields the equation

\[
(5 - 2 - 7) + k(8 + 3 - 15) = 0,
\]

i.e. \(k = -1\). Thus the desired equation is \((5x - 2y - 7) - 1(8x + 3y - 15) = 0\), which simplifies to \(3x + 5y = 8\).

Thus the conjecture holds true for this example.
Discussion:
In general assume that we have the system
\[
\begin{align*}
A_1 x + B_1 y &= C_1 \\
A_2 x + B_2 y &= C_2
\end{align*}
\]
and that \((r, s)\) is the external point. Since the desired line passes through the point \((r, s)\) external to both lines, this point must satisfy the equation of the desired line. Therefore, the value of \(k\) is obtained by substituting \((r, s)\) into equation (2) and solving for \(k\). Thus the value of \(k\) must equal \(\frac{-A_1 r + B_1 s - C_1}{A_2 r + B_2 s - C_2}\).

Theorem 3
The equation of a line passing through the point of intersection of two lines \(A_1 x + B_1 y = C_1\) and \(A_2 x + B_2 y = C_2\), and the point \((a, b)\) external to both lines is:
\[
(A_1 x + B_1 y - C_1) + k (A_2 x + B_2 y - C_2) = 0,
\]
where \(k = \frac{-A_1 a + B_1 b - C_1}{A_2 a + B_2 b - C_2}\).

Proof: Consider the equation
\[
(A_1 x + B_1 y - C_1) + k (A_2 x + B_2 y - C_2) = 0.
\]
Let us substitute the value of \(k\) and then simplify to obtain
\[
(A_1 x + B_1 y - C_1)(A_2 a + B_2 b - C_2)
\]
\[
-(A_2 x + B_2 y - C_2)(A_1 a + B_1 b - C_1) = 0.
\]
Collecting the coefficients of \(x, y\), and the constant term, we get
\[
(a A_1 A_2 + b A_1 B_2 - A_1 C_2 - a A_1 A_2 - b A_1 B_1 + A_1 C_1) x
\]
\[
+(a A_2 B_1 + b B_2 B_2 - B_2 C_2 - a A_2 B_1 - b B_2 B_1 + B_2 C_2) y
\]
\[
-a A_1 C_1 - b B_1 C_1 + C_1 C_1 + a A_1 C_2 + b B_2 C_2 - C_2 C_2 = 0.
\]
Simplifying the coefficients of \(x, y\), and the constant term, we obtain
\[
(b B_2 B_1 - A_1 C_2 - b A_2 B_1 + A_2 C_1) x
\]
\[
+(a A_1 B_1 - B_1 C_2 - a A_1 B_1 + B_1 C_1) y
\]
\[
-a A_2 C_1 - b B_2 C_1 + a A_1 C_2 + b B_2 C_2 = 0.
\]

We now find the equation of a line that passes through the point of intersection of the two lines \(A_1 x + B_1 y = C_1\) and \(A_2 x + B_2 y = C_2\), and the point \((a, b)\). Using Cramer’s rule, the point of intersection of the two lines is
\[
x = \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1}
\]
\[
y = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1}.
\]
Therefore the slope of the desired line is
\[
m = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1} - \frac{b}{A_2 B_2 - A_1 B_1}.
\]
The equation of the line passing through the point of intersection of the two lines and \((a, b)\) is
\[
y - b = \frac{(A_1 C_2 - A_2 C_1)}{(B_1 C_2 - B_2 C_1)} - \frac{b (A_2 B_2 - A_1 B_1)}{(B_1 C_2 - B_2 C_1)} (x - a)
\]
or
\[
(A_1 C_2 - A_2 C_1 - b A_2 B_2 + b A_1 B_1) x
\]
\[
-(B_1 C_2 - B_2 C_1 - a A_1 B_2 + a A_2 B_1) y
\]
\[
-a A_1 C_2 + a A_1 B_1 + ab A_2 B_2 - ab A_1 B_1
\]
\[
+b B_2 C_2 - a A_1 B_2 + ab A_2 B_1 = 0.
\]
Simplifying the constant term, multiplying both sides by \(-1\), and rearranging, we get equation (3). This completes the proof of Theorem 3.

In closing, this final theorem provides a different method to find the equation of a line through the intersection of two given lines and an external point. With a little practice students should find this to be the quickest way to solve this particular type of problem. The use of linear combinations as done in this final theorem is a prerequisite skill to the study of vectors and topic in advanced linear algebra. The ideas presented in this paper can help a student observe that there could be an
underlying common thread in seemingly unrelated mathematical concepts.

References


Watson and the Mysterious Number One – Solution
(from the article beginning on page 30)

\[
\begin{align*}
\text{n}^2 + (a - y)^2 &= y^2 \\
\frac{n^2 + a^2}{2a} &= y \\
\text{n}^2 + (2y - a)^2 &= (\sqrt{a+1})^2 \\
\text{n}^2 + \left[2 \left(\frac{n^2 + a^2}{2a}\right) - a\right]^2 &= a + 1 \\
\text{n}^2 + \left(\frac{n^2}{a}\right)^2 &= a + 1 \\
\text{n}^2 + \frac{n^4}{a^2} &= a + 1 \\
\text{a}^2 n^2 + n^4 &= a^3 + a^2 \\
\text{n}^4 + a^2 n^2 - (a^3 + a^2) &= 0 \\
\left(n^2 + (a^2 + a)\right)(n^2 - a) &= 0
\end{align*}
\]

Thus, \( n^2 + (a^2 + a) = 0 \) or \( n^2 - a = 0 \).

So, \( n^2 = -(a^2 + a) \) or \( n^2 = a \).

So, \( n = \sqrt{a} \).
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