THE ILLINOIS MATHEMATICS TEACHER

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Volume 56, No. 1 Spring, 2005

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From the Editors…

Welcome to the Spring issue of the *Illinois Mathematics Teacher*. We hope the school year is going well.

This issue of the *IMT* once again contains many useful and interesting articles and activities covering a wide range of topics. John Morrill wrote an article about Gertrude Hendrix, an ICTM charter member who is turning 100. "The Circles of Berit Wentworth" is written by a high school geometry class. Stan Izen encourages us to think of mathematics as a language. We also hope you enjoy the two reprints from an earlier IMT, one at the elementary level and one at the middle school level. Finally, we hope you have some time to work on the geometry crossword puzzle.

As usual, included in this issue is a form for becoming a reviewer for the *IMT*. In order to ensure that the articles are of interest to our readers, we send them to reviewers to get their approval. Also, there is a membership application form on the inside back cover.

We would appreciate hearing from any of you out there who are reading the journal. We would especially like to hear about any activities from the *IMT* that you have used with your students. Your comments and constructive criticism are heartily solicited. We are always especially eager to receive your submissions for publication. The guidelines are printed below.

Thank you for sharing.

Marilyn and Tammy editors

Guidelines for Submitting Manuscripts

Prospective authors should send:

- **Five (5) copies of your article**, typed, double-spaced, 1-inch margins. Put your name and address only in the cover letter. No identifying information should be contained in copies of the manuscript. Articles should be no more than ten pages in length, including any graphics or supplementary materials.
- **A diskette with your article, including any graphics.** We prefer that the article be written in Microsoft Word and that it be saved on an IBM-compatible disk. Graphics should be computer-generated or drawn in black ink and fit on an $8\frac{1}{2}'' \times 11''$ page.
- **Your name, address, phone, and e-mail** (if available) should be included in a cover letter.
- **A photo of yourself (Illinois authors only),** color or black/white.
- **Articles may be submitted electronically to tvoepel@siue.edu.**

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An ICTM Charter Member Enters Her 100th Year

A Celebration of a Teacher-Scholar's Life

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In 1964, the ICTM chose Gertrude Hendrix, Research Associate Professor at the University of Illinois, for its Distinguished Life Achievement in Mathematics Award. The criteria for this award require "distinguished and honorable service to mathematics education in Illinois." Her contributions include editing the ICTM News Letter for five years, serving as ICTM representative to the Illinois Curriculum Program, serving on the Executive Board and as Section Chair, and being a member of the joint ICTM-MAA Committee on the Strengthening of Mathematics for six years, including four as chair. One might also suspect that she was chosen because she is what Professor Zalman Usiskin of The University of Chicago defined as a pioneer, one of those people who make progress possible. In "The Stages of Change", a 1999 address about the major American mathematics reform movements, he said:

> In the case of the new math, the original pioneers were in a group called The University of Illinois Committee on School Mathematics, UICSM. There were three principals in that group: Gertrude Hendrix, a professor of education who wrote about what was called unverbalized awareness; Herbert Vaughn, a professor of mathematics who felt that if mathematics were made rigorous by the precise use of language and notation, then children would better be able to learn it; and the person who put it all together, Max Beberman, who believed fervently that one learned better if one was led to discover mathematics rather than being told it. (4, 5)

Gertrude Hendrix entered DePauw University in the fall of 1922. Since she knew, or thought she knew from the time that she was six years old, that she wanted to be a mathematics teacher, she was a bit dismayed to hear when she arrived that the head of the mathematics department was "very skeptical about women in mathematics." Years later she recalled that her reaction to that was "I've never been one to be deterred from doing anything I thought I could do simply because of the accident of having been born a girl, so I went ahead." She graduated, Phi Beta Kappa, in 1926 with a major in mathematics and a minor in physics. She spent her first two years after graduation teaching at a small high school in southern Illinois, and the next two years at The University of Illinois where she earned an M.S. in Education and taught at the University High School. In 1930 she went to Eastern Illinois University where she would give long and distinguished service. It was there that she had her first opportunity to learn to ride horses. Eventually riding, raising and studying horses would become an extremely important part of her life. In fact, a part intimately connected to her life's major research interest which she once described as the study of "the nature of unverbalized awareness traced from primitive animal learning to abstract human thought." In 1935 Gertrude Hendrix earned a second masters degree from Illinois, an

M.A. in mathematics. She also spent two summers and the 1946-1947 academic year taking graduate courses in philosophy, mathematics and psychology at The University of Chicago.

For twenty-six years at Eastern Illinois, she taught, trained teachers, conducted her research and served in leadership roles in various professional organizations. By the time Max Beberman asked her to join the UICSM group as a senior member, she had a firmly established professional reputation. She later said "Believe me, that decision was an upheaval, but I knew it was the most significant thing in school mathematics that had come along in my lifetime, and I was thrilled to join the staff."

What is this notion of "unverbalized awareness" which has been the focus of Hendrix's research? We will start by examining a portion of Beberman's 1958 Inglis Lecture at Harvard. In it he described the UICSM new math as being built on two principles – discovery and precision of language. In discussing the notion of discovery he said

> It is important to point out here that it is unnecessary to require a student to verbalize his discovery to determine whether he is aware of the rule. The teacher can use a sequence of questions to determine whether the awareness is present. In fact, immediate verbalization has the obvious disadvantage of giving the game away to other students, as well as the more serious disadvantage of compelling a student to make a statement when he may not have the linguistic capacity to do so. … This technique of *delaying* the verbalization of important discoveries is characteristic of the UICSM program, and differentiates our discovery method from other methods which are also called 'discovery methods' but which always involve the immediate verbalization of discoveries. (26,27)

The word technique in the quote above has a footnote. The reader is referred to Gertrude Hendrix's 1947 paper "A New Clue to Transfer of Training." This paper is, in fact, the only reference cited in the entire section of the lecture dealing with discovery. The article was written during her 1946-1947 sabbatical year at Chicago where she conducted the experiments testing her belief that there was an unverbalized awareness stage in learning by induction, and that the learner could do something with a generalization even before it has emerged from the unverbalized form to a conscious level. Here are some of her own words from that paper:

> Three hypotheses that are rather startling in relation to the theory of instruction are suggested by the results of a recent experiment in educational psychology. The problem, on which the writer has been working more or less informally for nearly ten years, is stated as follows: To what extent, if any, does the way one learns a generalization affect the probability of his recognizing a chance to use it? (197)

Her experiment led her to say

> Hypotheses emerging from the data are:
>
> 1. For generation of transfer power, the unverbalized awareness method of learning a generalization is better than a method in which an authoritative statement of the generalization comes first.

2. Verbalizing a generalization immediately after discovery does not increase transfer power.
3. Verbalizing a generalization immediately after discovery may actually decrease transfer power. (198)

Not to belabor the issue, but a few more of Hendrix's words. After all, this exercise is a celebration of her life and work.

> "Yes. What is the next one after that?" "Twenty-sev – No! Twenty `-nine" He had it. Very soon almost every member of the class seemed to be finding the next prime independently. Something subverbal – something organic, if you please – had happened to them. … They had the prerequisite to meaning of the term [prime]. …I would like to call that sub-verbal, organic, dynamic state of awareness the possession of the concept, but most psychologists, linguists and philosophers today do not consider the concept complete in the person's mind until he attached a symbol to it. The failure to recognize that awareness of an entity is independent of the existence of a symbol for the entity, promotes pegagogy that is not only wasteful, but often harmful. (1950, 334)

> Critical Unanswered Questions
> 1. Why is something which has come to a person first as insight (i.e., unverbalized awareness) much more likely to "pop into his mind" when he needs it than something which he has acquired through interpreting a linguistic formulation? … 4. Under what conditions does attempting to verbalize (that is, name and define) a concept immediately after one has attained it mutilate the concept in his own mind? … 8. For a given individual, how can one know when it is not too soon for him to verbalize a discovery? (1960, 58, 59)

> The issue of learning by discovery and teaching for learning by discovery is beclouded at present by the fact that each three very different procedures is being called 'The Discovery Method.' They are the inductive method, the nonverbal awareness method, and the incidental method. … The inductive method is nothing new in mathematics education. Colburn's book on teaching arithmetic by this kind of approach was first published in the 1820's. … The fallacy in the inductive method lies in its confusion of verbalization of discovery with the advent of the discovery itself. (1961a, 290)

In this latter paper Hendrix dismisses the incidental method as "an approach widely promoted in the Progressive Education era"; an "activity program" "doomed to triviality." In this 1961 paper she also uses the word "multilate[d]", which appeared in the 1960 paper above, and in the same context. Here it has a footnote: "I use the word 'multilated' deliberately and almost angrily. My feeling about this issue has been built up from several years experience in seeing this thing happen to children over and over again." (297) This footnote is just a bit of the available evidence of how passionately Gertrude Hendrix felt about finding ways for children to best learn.

There are others who have looked at this notion of what Hendrix called unverbalized awareness, including Schwartz (1948), Wills (1967) and Haslerud and

Meyers (1958), who have provided evidence for her position. Haslerud and Meyers, in fact, claim that "[Their] results give strong support to the postulate of Hendrix …" (297) There are others who are skeptical. For example, Sowder (1974) says "[his] study does not support the Hendrix hypothesis with respect to short-term retention … " (175) He does suggest that "further consideration of this problem is warranted" (175) However, there is not much doubt that the "Hendrix Hypotheses" of 1947 continues to be of interest to some researchers in the field. In November of 2003 an e-mail communication to me from Paul Goldenberg at the EDC Center for Mathematics Education included "Gertrude's idea about the risks of too-early verbalization is generating quite a lot of current conversation. Despite the age of this research, its import is quite current, as asking students to explain mathematical ideas and processes orally and in writing is very much in vogue today."

Gertrude Hendrix spent ten years with the UICSM project working with teachers, experimenting with curriculum, conducting training institutes and lecturing throughout the country. In addition, she was the content director for fifty UICSM teacher training films, a project for which Hendrix recruited Margaret Mead, the noted anthropologist, as consultant. Even with all of these responsibilities, she continued her research involving unverbalized awareness and the nature of language and communication.

When she left Illinois, she returned to land in Indiana which had been in her family since 1829. Polycreek Farm became her home, and the letterhead of her stationery told you it dealt in hardwood timber, Black Angus cattle and American Saddlebred horses. She did keep her hand in mathematics education, serving for a time as a consultant to the UICSM, the DePauw University mathematics department and to the Greencastle Community Schools. She also continued her research, produced a major paper in 1968, and in 1988 published her last work, Nature of Language, which appeared a mere fifty-six years after her first publication.

Gertrude Hendrix's written legacy includes two books, over twenty mathematical articles, and several pieces dealing with horsemanship and training. She was listed in several editions of Leaders in American Science, was a member of Phi Beta Kappa, Kappa Delta Pi, Pi Mu Epsilon, and Kappa Mu Epsilon, and received an Alumni Citation from DePauw. She was never listed in Who's Who of American Women. She was, however, asked several times, and her reply to one request includes "I do not wish recognition from any agency which classifies human beings into men and women before it classifies them according to achievement. The implication of Who's Who of American Women is that a woman must not be expected to do as much as a man to achieve distinction. I find this degrading, almost insulting, and I have never comprehended why all self-respecting women do not feel the same way."

Gertrude Hendrix lives in The Asbury Towers Retirement Community in Greencastle, Indiana. The party to celebrate her 100th birthday will be held there on May 24, 2005.

Selected Publications of Gertrude Hendrix

(1936) A Protest Against Informal Reasoning as an Approach to Demonstrative Geometry. The Mathematics Teacher 29, 178-180.

(1937) Plane Geometry and Its Reasoning, New York, Harcourt Brace (with Harry C. Barber.)

(1941) What Mathematical Knowledge and Abilities for the Teacher of Geometry Should Teacher Training Programs Provide in Fields other than Geometry. The Mathematics Teacher 34, 66-71.

(1947) A New Clue to Transfer of Training. The Elementary School Journal 48, 197-208.

(1950) Prerequisite to Meaning. The Mathematics Teacher 43, 334-339.

(1955) Developing a Logical Concept in Elementary Mathematics. The Mathematics Teacher 48, 542-547.

(1959) Variable Paradox. School Science and Mathematics 59, 461-464.

(1960) Nonverbal Awareness in the Learning of Mathematics. Cooperative Research Monograph No. 3: Research Problems in Mathematics Education, Washington, D.C., U.S. Government Printing Office, 57-61.

(1961a) Learning by Discovery. The Mathematics Teacher 54, 290-299.

(1961b) The Psychological Appeal of Deductive Proof. The Mathematics Teacher 54, 515-520.

(1968) Unverbalized Awareness as an Agency for Transfer of Learning. an invited paper presented to a symposium, The Effects of Conscious Purpose on Human Adaptation, sponsored by the Wenner-Gren Foundation for Anthropological Research, Burg Wartenstein, Austria (reprinted in Nature of Language, 7-25.)

(1988) Nature of Language, Charleston, Eastern Illinois University.

Other References

Beberman, Max. (1958). An Emerging Program of Secondary School Mathematics, Cambridge, Massachusetts, Harvard University Press.

Haslerud, G.M., and Myers, Shirley. (1958). The Transfer Value of Given and Individually Derived Principles. The Journal of Educational Psychology 49(5), 293-298.

Schwartz, B.D. (1948). Certain relations between verbalization and concept formation. Doctoral dissertation, Princeton University.

Sowder, Larry. (1974). The Influence of Discovered Numerical – or Sorting-Task Generalizations on Short-term Retention in Connection with the Hendrix Hypothesis. The Journal for Research in Mathematics Education 5, 167-176.

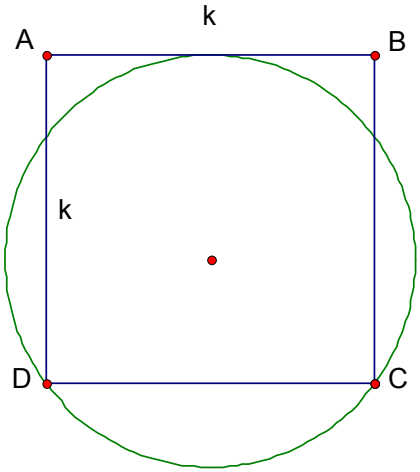
Usiskin, Zalman. (January, 1999). The Stages of Change. Available: http://lsc-net.terc.edu/do.cfm/ conference_material/6857/show/use_ set-oth_pres.

Wills, Herbert. (1967). Transfer of Problem Solving Ability Gained Through Learning by Discovery. Doctoral Dissertation, University of Illinois.

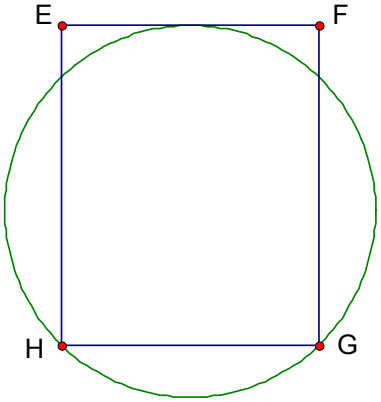
The Circles of Berit Wentworth

Andrew Samide's 1st period Geometry Class
Wheaton North High School
Wheaton, IL

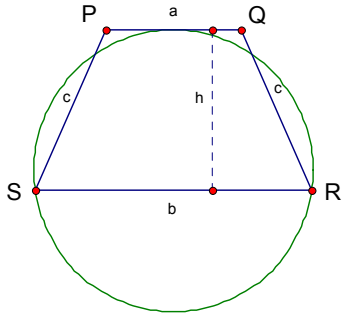
It was your average Wednesday morning, first period. The students in the geometry class were trying to wake up for another day of school. The freshmen and sophomores had been working on a tough problem and finally found a solution. The problem dealt with a square and a circle that was drawn through two of the square's vertices and tangent to one of its sides (as shown).



Their challenge was to find the radius of the circle. The class of students was relieved to have solved this problem. They found the solution was 5/8 k, k being a side length of the square. However, in this teacher's class, students are praised for thinking outside the box and coming up with new problems that are formed from the original one. First, they placed the circle in a rectangle that would replace the square. The pupils found that this problem was easily solved because of the information they had acquired from the previous one.



So the teacher asked the class if anyone else could think of another shape they could place the circle in. One student spoke up and suggested placing the circle in an isosceles trapezoid. The instructor and his class loved the idea, but knew that they had a challenge ahead of them. The teacher quickly put together a diagram (as shown) and the students got to work.



The problem was to find the radius of a circle that is tangent to the smaller base of an isosceles trapezoid and contains the endpoints of the longer base. The class tried a number of approaches over several days but remained stumped by the problem's complexity. On the third day of working with this problem, one student finally found

a successful method of solving it. He demonstrated a number of steps he used to solve it. They were as follows:

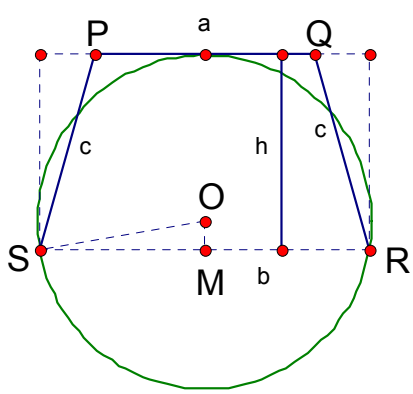
$$(OS)^2 = (OM)^2 + (MS)^2$$

$$r^2 = (h-r)^2 + \left(\frac{b}{2}\right)^2$$

$$r^2 = h^2 - 2hr + r^2 + \frac{b^2}{4}$$

$$2hr = h^2 + \frac{b^2}{4}$$

$$r = \frac{(4h^2 + b^2)}{8h}$$



OS = r

MS = b/2

OM = h - r

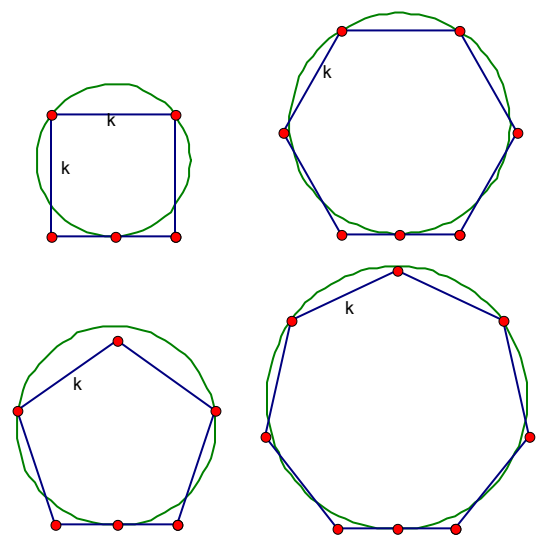
In solving this problem, the class learned how to form one problem from the original by carefully studying it. After further examination of the original problem, the class realized that several other similar problems could be created. The problems that follow demonstrate other types of problems that have been created by the students from the original one.

1. Given an isosceles trapezoid. Construct two Berit Wentworth circles.

Berit Wentworth circles:
   a. the circle that contains the endpoints of the longer base and tangent to the smaller base.
   b. the circle that contains the endpoints of the smaller base and tangent to the longer base.

2. Construct an isosceles trapezoid so that each of the following circles can be constructed.
   a. Circumscribed Circle
   b. Inscribed Circle
   c. Berit Wentworth Circles

3. Given the four regular polygons as shown. Find the radius of each circle in terms of k. Generalize for any regular n-gon of side length k.



We hope your classes will enjoy investigating and solving the original problems as much as we did when presented to us in class. We also hope you can extend your learning and understanding of geometrical concepts by solving the problems suggested at the end of this article.

Back Row
Kelly Simon, Kelly Waterman, Jenny Kraakevik, Stephen Huff

Front Row
Tanya Pardungkiattisak, Rachel Burke

Not Pictured
Giovanni Partida, Leah Johnson, Brendan Taveirne, Jenny Cather, Nile Kurschinski, Matt Tytel, Sean Norris, Melissa O'Brien, Zach Creer, Katie Storm, Derek Ho, Todd Irvin

Mathematics Is A Language

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This is an exciting and challenging time to be a math teacher. Technology has opened the door to teaching and doing mathematics in new and different ways. Because of computers and graphing calculators, mathematics educators are in a continuous process of revising curriculum to take full advantage of available technology. The opportunity to create new curriculum carries with it the necessity to learn new methods and techniques as well as the responsibility to make correct choices. Implied in this process of reworking mathematics curriculum is a basic question: what above all else do we want to teach our students about mathematics? To me the answer is clear; I want my students to know that mathematics is fundamentally a language.

Webster's Unabridged Dictionary, 1996, offers the following as one definition of language: "any system of formalized symbols, signs, sounds, gestures, or the like used or conceived as a means of communicating thought, emotion, etc.: *the language of mathematics* . . . " Mathematics, with its symbols and operations, is a language that allows one to write mathematical sentences that convey relationships in the physical world. A second component of mathematics is the system of postulates and theorems that dictate how one manipulates those sentences. I believe that many, if not most, math teachers de-emphasize the first in favor of the second. Why is this the case? In part because creating accurate mathematical sentences is much more difficult than learning to solve those sentences, as anyone knows who has tried to teach students to solve word problems. Equations and inequalities tend to fall into categories that can be solved in similar ways while writing equations is usually a unique task.

Until now mathematics educators have chosen to devote much of the early years of math curriculum to skill building, leaving the representational aspect of mathematics until high school, college, or perhaps, never. But technology has given us better ways to do computations and many algebraic operations, freeing us to devote more time to understanding concepts and teaching how mathematics can model the real world. Mathematics is much more than,

- finding common denominators,
- simplifying radicals,
- factoring polynomials,
- solving systems of three or more linear equations by hand,
- approximating the irrational roots of a polynomial equation by hand, using the Intermediate Value Theorem.

Perhaps math teachers have delayed teaching the descriptive ability of mathematics, thinking that thorough facility with skills is necessary to make sense of this aspect of the subject. I think that a good case can be made for just the opposite. What really makes mathematics difficult for many students is being asked to move symbols around without understanding the meaning of the symbols or the larger

purpose for which the work is being done. It is like trying to translate a sentence written in a foreign language while knowing only a little of the vocabulary. Unless we relate what we are doing to real situations, unless there is a context for solving equations and graphing functions, we are missing the point of mathematics and we are shortchanging our students. Professional mathematicians are comfortable with abstraction; they understand how mathematics is connected to the world and have moved beyond it. Students, on the other hand, do not have the sophistication necessary to manipulate symbols that are, apparently, unrelated to anything, that have no understood meaning. Mathematics needs to be tethered to reality if we are to expect students to connect with it. Fortunately, technology can save us from the tyranny of excessive computations if we are insightful enough to realize that mathematics is more than routine calculations.

When someone reads an article in the newspaper about an exploding population or growth in inflation somewhere in the world, she should think – exponential function. She should be able to picture the graph in her mind and think about its domain and range and what they say about population or inflation. If one sees a curved overpass and by chance wonders how tall a truck will safely drive through, she should know that the maximum height and the width of the overpass can yield the equation of an ellipse which can then be used to answer the question. When someone is discussing how quickly a NASA rocket is lifting off, she should know that a derivative will describe the rate at which the rocket is moving at any given moment. When one hears "the length is twenty feet more than the width" she should think to herself, x and $x + 20$.

Just as ordinary language gives us a vocabulary and a syntax to systematize everyday thoughts, mathematics organizes ideas relating to science, business, and more. This is what scientists do every day. Formulas and equations, created from data, are used to describe various phenomena and to make sense of the world around us. These formulas can be utilized to interpolate and/or extrapolate new data relating to events that are too large, too small, or too difficult to recreate in a laboratory as well as to predict future behavior. Function is the key concept that one uses to translate the world into mathematical sentences. Linear functions model situations that feature constant rate of change. A simple linear function, such as $f(x) = 0.15x + 36$, can be interpreted as saying, "the initial amount of f is 36 and its rate of change is 0.15." Thus, one could use this sentence, for example, to represent the cost billed by a cell phone company that charges a base fee per month of \$36 plus 15¢ per minute. Sinusoidal functions are used to represent periodic relationships, such as tides and simple harmonic motion. Any harmonic motion can be described by the equation $f(t) = a\sin b(t + c)$.

Understanding mathematics to be a language, and teaching it from that viewpoint requires a shift in how one teaches. Largely this is a change in how one thinks about mathematics that requires one to always ask, "what does this equation or expression *say* about the physical world?" For example,

- Instead of thinking of a function only as a set of ordered pairs, one would emphasize the relationship that creates the ordered pairs.

- Instead of repetitious exercises computing function values such as $f(2)$ and $f(t+1)$ when $f(t) = -16t^2 + 160t$, one can also interpret this expression in light of the real world meaning of the function f, e.g. $f(t)$ is the height of a

rocket t seconds after being launched from ground level with an initial velocity of 160 ft./sec.

- In addition to solving equations, one can ask students to write a situation that can be modeled by a given sentence.

- Instead of spending a week (or more) learning how to simplify radicals, a skill of doubtful value in the calculator age, one can discuss how $y = \sqrt{x}$ and $P = P_0 e^{kt}$ model different kinds of growth. For example, Halley's Law, $p(x) = 29.92e^{-0.2x}$ for $x > 0$, gives barometric pressure, $p(x)$, as a function of altitude, x miles, above sea level.

- Instead of an assignment that asks students just to graph a number of functions, a teacher can ask students to make up real world situations that are modeled by given graphs.

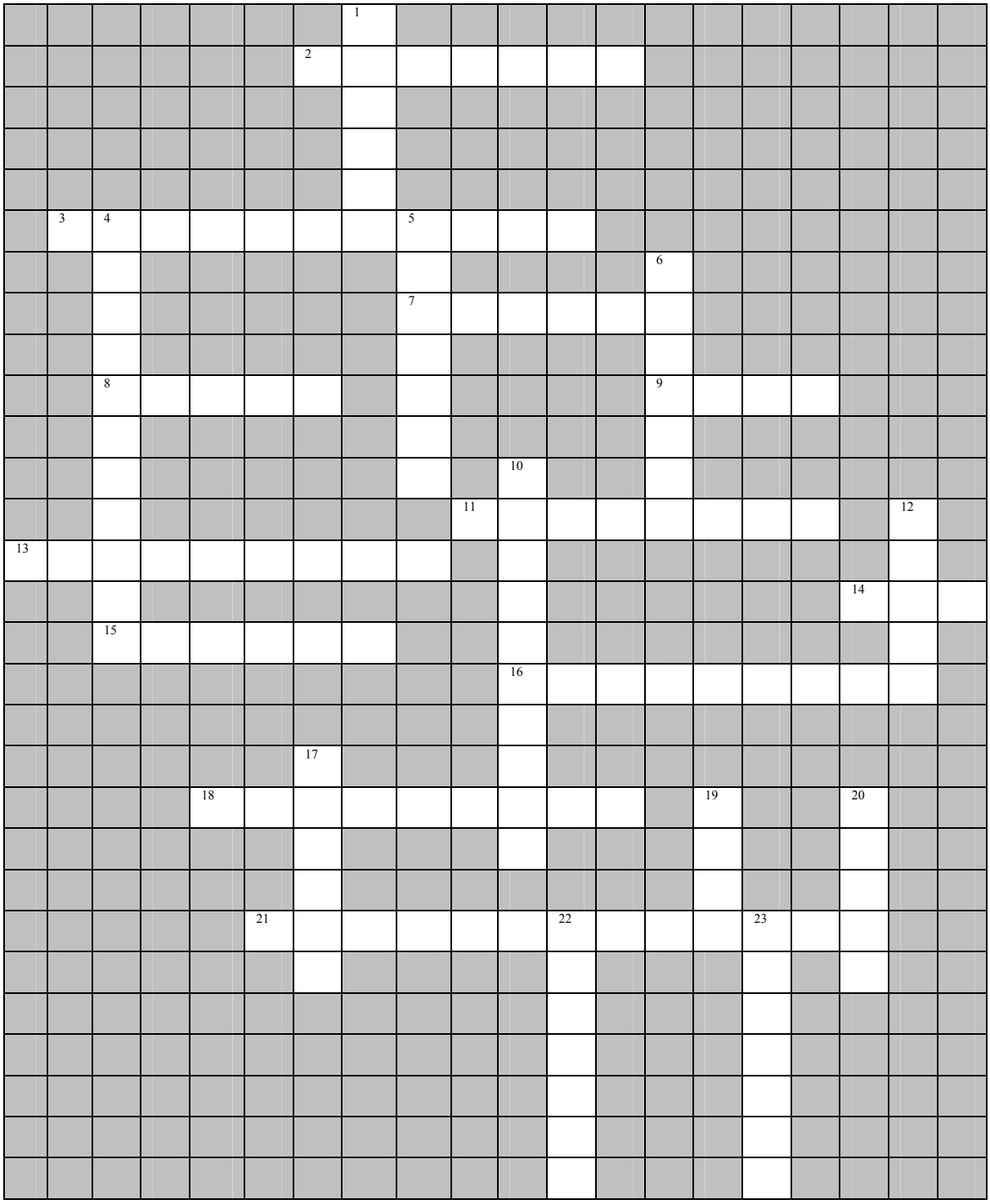
There are many advantages to emphasizing that mathematics is a language. First, mathematics becomes more meaningful to students because its link to the real world is stressed. To a large extent it answers the commonly heard and under-appreciated question, "Why are we learning this?" Second, from a pedagogical point of view, one can cover more topics and in greater depth because less time is spent on drill and practice. Less drill and practice also means less boredom for students. Third, and most important, students leave school with a powerful analytical tool, the ability to use mathematics to make sense of the world.

Certainly solving equations, graphing functions, and most mathematical skills are essential and need to continue to be taught thoroughly. But we cannot fall into the trap of thinking that this is all that mathematics is. The use of technology frees us from the drudgery of unnecessary manipulations and allows us the time to teach our students to see the world through mathematical eyes.

We've got our fingers crossed that you will lend helping hands by writing up some of the great ideas you have about teaching math and submitting them to the IMT

Thanks , ! See page 2 for how to submit..... *please don't forget* !

Geometry Crossword Puzzle



Geometry Crossword Puzzle
Clues

ACROSS

2. Used to construct circles.
3. Four sided, three-dimensional figure.
7. Set of points in a plane equidistant from a given point.
8. Not small.
9. In a rectangle, it is length times width.
11. Three sided plane figure.
13. Distance around a two-dimensional figure.
14. A straight line extending in one direction from a point.
15. One-dimensional measurement.
16. Parallelogram with one right angle.
18. Quadrilateral with one set of parallel sides.
21. Four sided, two-dimensional figure.

DOWN

1. Three-dimensional measurement.
4. Triangle with three congruent sides.
5. Ten-sided polygon.
6. Six-sided polygon.
10. Equality involving two ratios.
12. Point, line, __?__.
17. Half the diameter of a circle.
19. Hexahedron.
20. Union of two rays with a common endpoint.
22. ___?___ surface area.
23. Parallelogram with four congruent sides.

Geometry Crossword Puzzle - Solutions

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Developing a Mathematician's Mind by Requiring Reasoning and Pattern Finding

Cathy Kaduk
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Naperville, IL

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Mathematics, as the study of patterns and the study of numbers, shapes and symbols, provides a way to look at the elegance and beauty of our world. Richard Feynman, a well known American physicist, stated, "To those who do not know Mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature..." (Feynman, 1994). This article provides examples of students role-playing as mathematicians in problem-solving settings. Through these tasks children gain insight into the elegance and beauty of our world while developing mathematical reasoning and pattern-finding abilities. The atmosphere of students doing a task, or solving a "big" problem, has a sense of excitement in which students develop higher-level thinking and communication skills. (Bright,1996; Chancellor,1991; Kamii,1993; Stanic,1990).

In our district, students who show academic promise have an opportunity to work in small groups as part of an enrichment program. While the examples described below were done as part of an enrichment program, they are adaptable to other classroom settings. Teaching students to reason, problem-solve, and communicate in the manner of a mathematician or engineer is for every student! Mathematics enrichment lessons fall into three main categories: students doing an extension of a math lesson, often connected to story problems, students doing an interdisciplinary task which involves concepts related to the classroom curriculum, and students trying to answer an "imponderable problem."

Story Problems

Story problems are an essential part of the extensions to the classroom curriculum. The story problem can pose a scenario where students apply prior learning and practice mathematical thinking via the questioning process. The four phases initially suggested by the well known mathematician George Polya are used as a basis. These are enhanced with the following 4 questions (numbered below) developed by Art Hyde (Hyde, 1991).

Polya's Steps
with Hyde's Questions

* Understanding the Problem.
  1. *What do you know for sure?*
  2. *What do you want to find out?*
  3. *Are there any special rules or conditions?*
* Devising a plan of attack
* Carrying it out
* Reviewing / Looking Back
  4. *Is there another way to do this?*

This process is modeled by our enrichment aides who work with the students. As students internalize the process, there is less and less modeling. In kindergarten and first grade, many of these problems are done directly with the enrichment aide. Since many

of these students' thinking skills go beyond their reading skills, the assistance of a fluent reader, is also a factor in the type of groupings we plan. In second and third grade, students will usually be in pairs to solve problems and come together to discuss their answers.

In looking at "What do you know for sure?" questions of vocabulary can come out as well as identifying layers of meaning for various facts. In answering the question, "What are you trying to find out?" students can learn how to analyze patterns and identify relationships. Several kinds of practice are used to provide students with strategies to formulate questions. Alongside more traditional math story problems, a series of stories in the *Real Math Thinking Story Book* series by Stephen Willoughby and Carl Bereiter (1985) is used to teach reasoning skills. In these stories, students are to anticipate the questions a designated character will ask. Children develop their ability to reason through the text situation to form the questions. These are more teacher-guided in format, but have lots of opportunities for student input. These stories guide students to expand their thinking to include more possible questions and answer with specifics. Students learn to think about the dimensions of a problem or attributes of an object as they problem solve.

Students do like variety in the problem solving they do. They like story problems that involve familiar people, places or things. Meaningful text is an appealing part of what we term "Aunt Mathilda" problems, a valuable source of stories for developing reasoning (Andrews, 1996). These present situations with an embedded problem to solve and can be found on the internet (http://www.dcmrats.org/auntymath.html). Our students organize the story details so as to know the key question to answer and the information given. We like these Aunty Math problems because the story setting is like a conversation students might hear. Solving the problems from "Aunt Mathilda" might involve classic problems, ones with fractions, or combination problems, written into a story of interest to elementary age students, where the conditions must be extracted from the text.

For example, after reading "Gina's Pet Challenge–Challenge 31" students ponder how many cats and birds are in Gina's big family:

> "In my big family, we have 7 pets. Some are cats. The rest are birds. If you count the legs, you will count 20 legs. How many cats and birds do we have in our big family?"

Students read the story until they have a sense of the problem to solve and then how to do it. We encourage them to record what they know, then read for a sense of the problem to solve, then read looking for any special twists that make this problem unique. Students learn to record information step-by-step as they read the problem the second or third time through. In discussing, "What are you trying to find out?" and "Are there any special rules and conditions?" the need to use strategies such as the acting-it-out strategy, manipulatives, pictures, or using charts teaches students ways to organize what is known. The instructor gives guidance, but it is the student who does the thinking, drawing, and sometimes the writing, depending on circumstances. It is stressed that students should work in such a way that they can "prove" their answer is correct and that their thinking is sensible. After solving the problem, students are asked to substantiate their answers as well.

For example, in the Pet Challenge problem, students recorded what they knew and developed a plan. Most identified that the

number of animals was key information and kept track of the head count. Some made 7 birds, and then noted the need for more legs when checking to see if the solution met the requirements of the problem. Some made 7 cats and then realized there were too many legs to fit the conditions, and added birds to modify. A couple of children randomly drew cats and birds and kept a running total of the legs and heads. It was as if they were looking at the interactions of cats and birds as part of the criteria for the problem. Some children used stick figures, some use elaborate drawings and some made a chart showing combinations of heads and legs of birds and cats. Students who have checked their answer against the conditions of the problem seem more assured and confident about their work.

Afterwards, sometimes students ask, "Are there other problems like this we could solve?" Learning to make up problems for each other becomes the next step. Students self-extend the lesson, become more independent learners and refine within themselves the "research skills" of mathematicians as they record ideas, give evidence for their views, and find new information. In solving each other's problems, students note that sometimes they cannot really solve their friend's problem without more information or making an assumption. Furthermore, students' reading comprehension on word problems seems to improve.

A student teacher observing the students remarked at how the group's discourse was so different from what she would have anticipated. As the students compared their answers, looks of reflection and "hmm" sounds were observed. Instead of hiding the answer to the problem because it looked different, the student teacher noticed students asking each other, "How did you get that?" or "Why do you think so?" The observations surprised her because her past experience included students getting defensive if their answer did not look like their peers. She noted a sense of organization and purpose within their discussions.

Interdisciplinary Tasks

Students also come to enrichment to do interdisciplinary tasks which involve concepts related to the classroom curriculum. Science, social studies and mathematics can be combined as a situation unfolds. We have found that the TIMS (Teaching Integrated Math and Science) and AIMS (Activities Integrating Mathematics and Science) materials are good sources. Students have a question to answer, such as, Which ball bounces the highest when dropped?

Our third graders explored the regularity of Native American designs. For example, blanket border patterns by graphing the number of stencils versus the length of the border covered by the stencils. Materials from TIMS (Beissinger and Kelso, 1995), and the NCTM Math Notes (Barkley, 1997) were combined for the project. Stencils of the design are made with graph paper. Studies of Native American designs in social studies allowed students to come up with some authentic looking patterns. After applying the stencil to a brown paper bag, students measure the length of the pattern after using the stencil once, twice, three times, etc. Student work is taped together to make paper blankets. Looking at the stencil and the resulting design made using the stencil, small groups form hypotheses about the relationship between the length of the design, the length of the stencil and the number of times the stencil pattern is repeated. Students then graph the relationship between the stencil and the design on the brown paper to check their hypothesis. This can be quite exciting. Using the journal, or a

group log on chart paper, students record their understanding of the connection of the stencil length and total length of the design. "Why did it do this?" students want to know. Several students then noticed the connection between the numbers on the Y-axis of the graph and the table. "What are you trying to find out?" took on a new meaning with connection between the numbers and the line. The relationship students see from a repeated pattern and resulting lengths became graphic–it formed a line on the graph. Looking at each other's data, students are able to make sense of what might be creating the line – it was the consistency of the added amount. Students are able to determine that without the consistency of the repeated addition, the line will not form. Students made a new connection between graphs, repeated addition and multiplication.

Imponderables

Imponderable problems are the last focus for enrichment lessons in this article. Students are asked to think about problems that are often considered classics. For example, we used the classic boatman across the river problem:

> The Showman, His Tiger,
> a Duck, and the Sack of Corn
>
> "How good are you at thinking logically? This is an old problem. You need to help the showman cross a river with his tiger, duck and sack of corn. There is a boat but it is so small that it can only hold the showman and one of the others. The showman can't leave the tiger with the duck as the duck will be eaten. He can't leave the duck with the corn for fear of losing all the corn. How does he get across?" (King, 1999)

Students were asked how to solve this classic logic problem. They were initially provided with a cardboard boat, a blue paper river and puppets. When pairs of students began working on solutions, they used paper pictures of these things. Asking students, "What do you know about the problem?" made students reread to determine the conditions for safe transport. The quick thinkers immediately wanted to have the boat return to the other shore without anyone rowing it, but others pointed out the lack of logic to the idea. Once the fact that there might only be one item brought across at a time, students began working on possible orders for the tiger, duck and corn. In asking, "What is really needed to solve the problem?" another key fact became apparent: there might have to be animals returning to the initial shore to keep the three items in safety–the goal was to get all items on the other shore at the end of the boat ride - not just get them to the other side and not think about them anymore. A key point is finally clarified: what is the meaning of "can't leave" in the problem? One pair of students decided that if the showman is part of the group, the duck won't eat the corn nor will the tiger eat the duck. Which item has to go first? After another huddle or two, they now have a solution to share with the rest of the group. Students repeat the conditions of the problem and discuss the logic of their answer. They do this by confirming the meanings of the vocabulary of interest in the problem, such as "can't leave" and test out the conditions of the problem to check validity of the students' answer. The enrichment aide is available to point out flaws in logic or definitions and help the group know when an acceptable amount of consensus has been reached for that given problem.

Standards for behavior and quality of work are set. The goal is to make the students

responsible for communicating properly, giving the students tools to work together, rather than having the teacher impose a structure. Students cannot put down each other or themselves when stating they do not agree with someone else's answer. The focus is on the facts, not the person. Students are encouraged to take risks in sharing ideas. Mathematicians don't generally have a structure imposed on them when working together; they have to determine what will work for the people involved. Being a successful mathematician includes being flexible in thinking, yet able to verify that conditions of the problem have been met. Students are encouraged to be flexible in their thinking, and encouraged to verify that the conditions of the problem are met before stating they have the correct answer. Extending the problem to a different setting is another way students are thinking like mathematicians, who take solution systems from one setting to another to test validity of a theory and develop new understanding of the setting. Developing reasoning skills in children develops their skill as mathematicians.

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Monopoly Unit

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A week long "Monopoly Unit" was developed as a supplement to the "Exploring Data: Tables and Graphs" and "Working with Signed Numbers" sections in our two year Algebra textbook. A preliminary letter was sent home to parents informing them of the week's activities. During the first trial for this activity, students were asked to bring Monopoly sets to use in class. Subsequently, a classroom set of games was donated for such use. On the first day, students were randomly placed in groups of 5 or 6, given instructions for the duration of the unit, and advised about record keeping. They then played continuously for four class periods. To each group fell the choice of banker and/or property manager. Each group decided upon special rules of play, e.g., "Everyone must round the board before buying any property", or "All fine money goes into the Just Parking kitty." Each student was given a packet in which to keep the following records:

1. A five page check register in which to record all monetary transactions which occurred during the course of the game.

2. Four "End of Period" worksheets on which to record daily beginning and ending cash and property as well as his/her ending position on the game board.

3. An "End of Game" register on which to record cash, property, mortgaged property, etc. to determine net worth and thus, the winner.

On the fifth day students were given a quiz which focused on their ability to organize the data that they had accumulated over the past four days. They were asked to categorize all the money that was received(rental income, passing GO,...) and paid out (renting property, income tax,...). They were asked to create a bar graph and frequency distribution to represent their data. In addition, there were questions involving percent (Pay Income Tax-10%). Given transactions such as: "Won a Beauty Contest prize of $50" or "Pay each Player $20", students were asked to fill in a check register.

Observations from this week of activity:

1. Allow approximately 5-8 minutes each day to restart and end the game.

2. Allow approximately 12 minutes on the fourth day for clean-up and calculation of net worth.

3. In general, all students enjoyed the experience.

4. As expected, there were varying degrees of enthusiasm.

5. I feel that this was a worthwhile activity for my classes.

The check register, record keeping sheets, and quiz follow.

Date	#	Description	Amount	Deposit	Balance

Daily Record Keeping Worksheet

Day _____ Date _____

Starting Balance _____

Ending Balance _____

Property owned at the end of the period.

1. _____ Houses _____ Hotels _____
2. _____ Houses _____ Hotels _____
3. _____ Houses _____ Hotels _____
4. _____ Houses _____ Hotels _____
5. _____ Houses _____ Hotels _____
6. _____ Houses _____ Hotels _____
7. _____ Houses _____ Hotels _____
8. _____ Houses _____ Hotels _____
9. _____ Houses _____ Hotels _____
10. _____ Houses _____ Hotels _____
11. _____ Houses _____ Hotels _____
12. _____ Houses _____ Hotels _____

Ending the Game Worksheet

Name_____

To calculate your net worth

1. Cash on hand. 1. _____

2. Property owned – value on board. 2. _____

3. Mortgaged Property – ½ value on board. 3. _____

4. Houses and Hotels @ purchased price 4. _____
   (Hotels = purchased price + 3 houses)

Total _____

Quiz - Monopoly Unit

Name_____

1. What was your beginning balance on the first day that you played the game? 1. _____
2. What was your ending balance on the first day that you played the game? 2. _____
3. How much money did you collect from passing GO? 3. _____
4. To get out of jail you could roll doubles, use a Get-Out-of-Jail-Card, or pay... 4. _____
5. List the "categories" of expenses (money paid out). 5. _____
6. List the "categories" of income (money deposited). 6. _____
7. Fill out the check register given the following data.
   - 9/27 Received from bank <$1500>
   - 9/27 Bought Illinois Ave. <$250>
   - 9/27 Income Tax <$136>
   - 9/27 Rent from Illinois Ave. <$18>
   - 9/27 Purchase Hotel for Illinois Ave. <$320>

Date	#	Description	Amount	Deposit	Balance

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_____ (City)	_____ (City)
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MEMBERSHIP APPLICATION FORM

____ New Member ____ Reinstatement ____Renewal ____Change of Address

Name: _____

Check preferred mailing address, please complete both columns.

____ Home Address

_____

_____

_____

(City)

_____

(State) (Zip Code)

_____

(Phone Number)

_____

(E-mail address)

____ Work Address

_____

_____

_____

(City)

_____

(State) (Zip Code)

_____

(Phone Number)

_____

(E-mail address)

****

Profession (check one)

____ EC-3 Teacher
____ 4-6 Teacher
____ Jr. High/Middle Teacher
____ Sr. High Teacher
____ Special Education Teacher
____ Community College
____ College/University
____ Administration
____ Student
____ Retired
____ Institutional Member
____ Other _____

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____ Institutional Member, one year $70

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Math Contest Fund $_____
Total Enclosed: $_____

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